Rayleigh & Mie scattering cross section calculations and implications for weather radar observations

1 Learning Objectives

The following are the learning objectives for this assignment:

- Familiarize yourself with some basic concepts related to radio wave scattering, absorption, and radar cross sections
- Become acquainted with some of the fundamental definitions and foundations connected with the Mie theory
- Explore under what conditions the Rayleigh approximation can be applied and to what extent it accurately represents radio wave scatter and absorption
- Study the dependence of various radar cross sections on wavelength and temperature

2 Introduction

As electromagnetic radiation propagates though our atmosphere, it interacts with air molecules, dust particles, water vapor, rain, ice particles, insects, and a host of other entities. These interactions result primarily in the form of scatter and absorption, both of which are important for remote sensing studies of the atmosphere. Typically, the degree to which a "target" can scatter or absorb electromagnetic radiation is described through its cross section σ. When a target is illuminated by a wave having a power density (irradiance) given by S_i, it will scatter/absorb a portion of the wave. The cross section represents an apparent area, used to describe by what amount the radiation interacts with the target. An observer located at a particular location (described by θ and ϕ with respect to the wave's propagation vector) will be detect radiation scattered by the target with a power density given by by S_r. Assuming that the target has scattered the incident electromagnetic radiation *isotropically*, then the cross section σ can be directly calculated using

$$\sigma(\theta, \phi) = 4\pi r^2 \frac{S_r(\theta, \phi)}{S_i}, \tag{1}$$

where r is the distance between the target and the observer. In general, the scattering cross section will depend the angles θ and ϕ. That is, the scatter is not truly isotropic. Also note that the value of σ does not in general correspond to the geometric cross section of the target.

Here we will consider four different types of cross sections, which are commonly used in connection with radar. They are the scattering cross section σ_s, the extinction cross section σ_e, the absorption cross section σ_a, and the backscattering cross section σ_b. The scattering cross section multiplied by the power density of the incident wave is equivalent to total amount of energy removed from the electromagnetic wave due to scatter in all directions. A certain amount of the energy is absorbed, which results in a heating of the target. The amount of energy removed from the electromagnetic wave through this process is equal to the scattering cross section multiplied by the power density of the incident wave. The cumulative effective of scattering and absorption is described by the absorption cross section. Finally, if one considers the value of $\sigma(\theta, \phi)$ for scattered in the direction from which the wave originates (backscattered), then this value defines the backscattering cross section. See such books as **???** for additional explanation and clarification.

In 1908, the German physicist Gustav Mie formulated a complete scattering/absortion theory, which describes the interaction of electromagnetic waves with spherical dielectric particles [**?**]. He showed that a solution could be found given in terms of an an infinite series of electric and magnetic multipoles. According to Mie theory, the scattering, extinction, backscattering, and absorption cross sections can be expressed as

$$\sigma_s = \frac{2\pi a^2}{\alpha^2} \sum_{n=1}^{\infty} (2n+1)(|a_n|^2 + |b_n|^2) \tag{2}$$

$$\sigma_e = \frac{2\pi a^2}{\alpha^2} \sum_{n=1}^{\infty} (2n+1)[-\mathrm{Re}(a_n + b_n)] \tag{3}$$

$$\sigma_b = \frac{\pi a^2}{\alpha^2} \left| \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n - b_n) \right|^2, \tag{4}$$

and

$$\sigma_a = \sigma_e - \sigma_s, \tag{5}$$

respectively, where a is the drop radius and $\alpha = 2\pi a/\lambda$. The variable α is sometimes called the radio electric size or the normalized radius and allows us to explore scattering and absorption in terms of scaled quantities, which in turn are applicable for any radio-wave frequency. The coefficients a_n and b_n represent the magnetic and electric multipoles of order n, respectively. Note that the extinction cross section is the sum of the scattering and absorption cross sections. If only values for a_n and b_n containing α terms to fifth order or smaller are considered, then the following expressions remain

$$a_1 = -\frac{i}{45}\,(m^2 - 1)\,\alpha^5 \tag{6}$$

$$b_1 = -\frac{2\,i}{3}\,\left(\frac{m^2 - 1}{m^2 + 2}\right)\,\alpha^3\,\left[1 + \frac{3}{5}\,\left(\frac{m^2 - 2}{m^2 + 2}\right)\,\alpha^2\right] \tag{7}$$

$$b_2 = \frac{i}{15}\,\left(\frac{m^2 - 1}{2\,m^2 + 3}\right)\,\alpha^5 \tag{8}$$

In the Rayleigh approximation, only the lowest order contributions in α are considered. This is equivalent to setting all a_n and b_n to zero except b_1, which reduces to

$$b_1 = -\frac{2i}{3}\,\left(\frac{m^2 - 1}{m^2 + 2}\right)\alpha^3. \tag{9}$$

Quantity	Temperature ($^\circ C$)	Wavelength (cm)			
		10	3.21	1.24	0.62
n	20	8.88	8.14	6.15	4.44
	10	9.02	7.80	5.45	3.94
	0	8.99	7.14	4.75	3.45
	-8	—	6.48	4.15	3.10
κ	20	0.63	2.00	2.86	2.59
	10	0.90	2.44	2.90	2.37
	0	1.47	2.89	2.77	2.04
	-8	—	—	2.55	1.77
$\lvert K_m \rvert^2$	20	0.928	0.9275	0.9193	0.8926
	10	0.9313	0.9313	0.9152	0.8726
	0	0.9340	0.9340	0.9055	0.8312
	-8	—	—	0.8902	0.7921
Im$(-K_m)$	20	0.00474	0.01883	0.0471	0.0915
	10	0.00688	0.0247	0.0615	0.1142
	0	0.01102	0.0335	0.0807	0.1441
	-8	—	—	0.1036	0.1713

Table 1: Components of the complex refractive index along with values for $|K_m|^2$ and Im$(-K_m)$ as a function of radar wavelength and temperature for water. These data were taken from Table 4.1 of **?**. Dashed lines indicate that data are not available.

Therefore, we see that in the Rayleigh approximation, equations (**??**), (**??**), and (**??**) reduce to

$$\sigma_s = \frac{2}{3}\,\frac{\pi^2}{\lambda^4}\,|K_m|^2\,D^6 \tag{10}$$

$$\sigma_a = \frac{\pi^2}{\lambda}\,\text{Im}(-K_m)\,D^3 \tag{11}$$

$$\sigma_b = \frac{\pi^5}{\lambda^4}\,|K_m|^2\,D^6, \tag{12}$$

where the K_m is given as

$$K_m = \frac{m^2 - 1}{m^2 + 2} \tag{13}$$

and m is the complex refractive index expressed as $m = n - i\kappa$. The Rayleigh approximation applies when $D \leq \lambda/16$. This is equivalent to assuming that the water drop acts as a simple dipole antenna with a charge separation of $2\,a$.

Shown in the table below are values for the components of the complex refractive index m and the quantity K_m. These quantities are used when calculating the various radar cross sections for spherical water drops.

3 Hands-On Activities

We will now use the expressions of the absorption and scattering cross sections for the Rayleigh approximation and the values for Im(-K$_m$) and |K$_m$|^2 to examine the theoretical attenuation of radio waves in the presence of spherical liquid water drops for different temperatures and different radar wavelengths. Values for Im(-K$_m$) and |K$_m$|^2 can be found in Table **??**. In the following exercises, you will be asked to calculate the normalized cross section as a function of the normalized drop diameter. The normalized cross section is simply given in terms of the cross section σ as $4\,\sigma/(\pi\,D^2)$ and the normalized drop diameter is expressed in terms of the drop diameter D as $\pi\,D/\lambda$. By normalizing these quantities, it is easier to apply the results to any radar wavelength and to understand the cross sections in terms of the geometric size of the particles.

1. Create four separate plots of the normalized scattering cross sections (one plot for 20$^\circ C$, one for 10$^\circ C$, one for 0 $^\circ C$, and one for -8$^\circ C$) versus the normalized drop diameter at T = 0°. On each plot you will show the normalized cross section as a function of the normalized drop diameter for the radar wavelengths given by λ = 0.62, 1.24, 3.21, and 10 cm.

2. Repeat the process for the normalized absorption cross section. That is produce four plots (one for each temperature) on which you show the normalized cross section as a function of the normalized drop diameter for the four radar wavelengths (λ = 0.62, 1.24, 3.21, and 10 cm).

3. Repeat the process for the normalized extinction cross section. Recall that this will capture the additive contributions from scatter and absorption.

You can use *NaN* whenever the value is not defined (see excerpt of Matlab code). Compare your results for the case of T = 0°C with Figure 3.4 of **?**.

Next we examine the radar cross sections for the more general case using Mie theory. Two MATLAB functions are available at `http://www.ou.edu/radar/`.

`Mie.m`:	Used to calculate the scattering efficiencies
`Mie_abcd.m`:	Used to calculate the Mie coefficients (called by `Mie.m`)

The inputs of `Mie.m` are (m, x), where $m = n + i\kappa$ and $x = 2\pi a/\lambda$. Note that the convention for m used in the code is different than introduced above. The output is a nine element vector containing n, κ, x, $\sigma_e/(\pi a^2)$, $\sigma_s/(\pi a^2)$, $\sigma_a/(\pi a^2)$, $\sigma_b/(\pi a^2)$, and two other parameters that we have not discussed. That is, a call to the function would look like `result = Mie(m, x);` These routines were written by Christian Mätzler of the Institut für Angewandte Physik in Switzerland [**?**].

1. Use values for n and κ from the class handouts available on `desire2learn` to reproduce the left panel of Figure 3.3 of D&Z for $\lambda = 10\,$cm and $\lambda = 3.21\,$cm.

2. On a separate figure, create plots of the the full Mie expression for $\lambda = 10\,$cm and $\lambda = 3.21\,$cm at T = 0°.

3. Reproduce the steps in the second part, but this time also overlay on the figure your results from Problem 1 taken for T = 0° and λ = 3.21 and 10 cm (Rayleigh approximation).

References

Battan, L. J., 1973: *Radar observations of the atmosphere*. University of Chicago Presss.

Doviak, R. J., and D. S. Zrnić, 1993: *Doppler Radar and Weather Observations*. Academic Press, San Diego, 2nd edition.

Mätzler, C., 2002: MATLAB functions for Mie scattering and absorption, Version 2. IAP Research Report No. 2002-11, Institut für angewandte Physik, Universität Bern.

Mie, G., 1908: Beiträge zur Optik trüber Medien, speziell kolloidaler Metallösungen. *Ann. Physik*, **25**(3), 377–445.

Savageot, H., 1992: *Radar Meteorology*. Artech House, Boston.

Excerpt of MATLAB code used to find the cross sections in the Rayleigh approximation

```
% radar wavelength (mm);
lambda = [10 3.21 1.24 0.62]*10;
% temperature of water (C)
T_w = [20 10 0 -8];
% n from Battan Table 4.1
n_w = [8.88 8.14 6.15 4.44; ...
       9.02 7.80 5.45 3.94; ...
       8.99 7.14 4.75 3.45; ...
       NaN  6.48 4.15 3.10];
% k from Battan Table 4.1
k_w = [0.63 2.00 2.86 2.59; ...
       0.90 2.44 2.90 2.37; ...
       1.47 2.89 2.77 2.04; ...
       NaN  NaN  2.55 1.77];
% Km^2 from Battan Table 4.1
Km2_w = [0.928  0.9275 0.9193 0.8926; ...
 0.9313 0.9282 0.9152 0.8726; ...
 0.9340 0.9300 0.9055 0.8312; ...
 NaN    NaN    0.8902 0.7921];
% Im(-Km) from Battan Table 4.1
ImKm_w = [0.00474 0.01883 0.0471 0.0915; ...
  0.00688 0.0247  0.0615 0.1142; ...
  0.01102 0.0335  0.0807 0.1441; ...
  NaN     NaN     0.1036 0.1713];
```