Quantum Mechanics
Qualifying Exam – January 2026

Notes and Instructions:

- There are **5** problems and **6** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work**.

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^\dagger), \quad \left[\hat{a},\hat{a}^\dagger\right] = 1, \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle, \quad \text{and} \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\ . \end{aligned}$$

The Hermite polynomials:

$$\begin{aligned} H_0(y) &= 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2-2 \\ H_n(y) &= (-1)^n e^{y^2}\frac{\partial^n}{\partial y^n}e^{-y^2} \end{aligned}$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}} \qquad Y_2^{\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$Y_1^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} \qquad Y_2^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$$

Angular momentum raising and lowering operators:

$$\begin{aligned} L_\pm &= L_x \pm i\,L_y \\ L_+|\ell,m\rangle &= \hbar[\ell(\ell+1)-m(m+1)]^{1/2}|\ell,m+1\rangle \\ L_-|\ell,m\rangle &= \hbar[\ell(\ell+1)-m(m-1)]^{1/2}|\ell,m-1\rangle \end{aligned}$$

Gaussian Integral:

$$I_0(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = (\pi/\alpha)^{1/2}\ , \qquad \alpha > 0$$

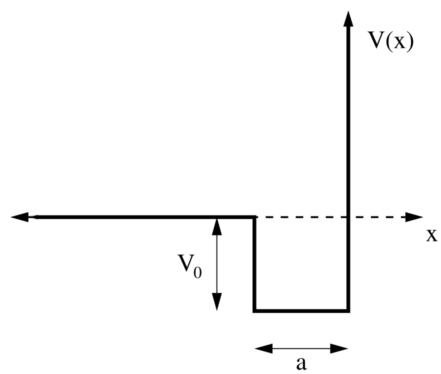
where α is usually chosen to be real.

PROBLEM 1: One Dimensional Potential Well

Consider a particle of mass m moving in one dimension subject to the potential:

$$V(x) = \begin{cases} 0 & \text{for } x < -a \\ -V_0 & \text{for } -a \le x < 0 \\ +\infty & \text{for } x \ge 0 \end{cases}$$

where V_0 and a are defined as positive constants.



(a) [3 points] First assume that the parameters are chosen so that there are exactly two bound states in the potential well. Sketch the probability density as a function of position for:

1. The ground state of the well.
2. The excited state of the well.
3. A scattering state for a plane wave incident from the left.

In each of your graphs label your axes clearly.

(b) [3 points] Now assume the potential well is "weak" so that V_0 and a can be arbitrary small, but not zero. Either prove that:

- There is always at least one bound state in the well, and calculate its energy as a function of the parameters given, *or*
- There might or might not be a bound state, and determine the requirement on the parameters for it to exist.

(c) [2 points] Lastly, assume that there is a plane wave of unit amplitude incident from the left, e^{ikx}. Find the steady state wave function in all space.

(d) [2 points] For this last scattering case, show that the maximum amplitude of the wave function in the well never exceeds the maximum amplitude outside the well.

PROBLEM 2: Simple Harmonic Oscillator

An electron is bound to a region of space by a simple harmonic oscillator potential

$$V(x) = \frac{1}{2}x^2$$

with an effective spring constant of $\kappa = 95.7\ \text{eV/nm}^2$. Given that the ground state wave function is

$$\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

where

$$\alpha = (km/\hbar^2)^{1/2}\,,\ \hbar c = 197.33\ \text{eV} \cdot \text{m}\,, \quad \text{and} \quad mc^2 = 0.5\ \text{MeV}\,.$$

(a) [1 point] What is its ground-state energy?

(b) [1 point] How much energy must be absorbed for the electron to jump one energy level?

(c) [3 points] If it is in the ground state, what is the probability to find the electron in a narrow interval of width 0.004 nm located halfway between the equilibrium position and the classical turning point?

(d) [3 points]]What is the expectation value of position uncertainty in the ground state?

(e) [2 points] Show that the position uncertainty calculated using the wave function is the same as what one would derive using the uncertainty principle.

PROBLEM 3: Identical Spin$-1/2$ Particles

(a) [3 points] Consider **two** nonrelativistic, noninteracting spin-$\frac{1}{2}$ fermions each of mass m in an infinite square well potential of length L between $x = -\frac{L}{2}$ and $x = \frac{L}{2}$. Find the five lowest energy eigenstates and eigenvalues of this system.

Hint: Express a two-particle eigenstate as a product of a spatial wave function and a spin wave function. Let $|n\rangle_i$ denote putting the $i-$th electron in the $n-$th eigenstate of the well and let $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ correspond to up and down spinors for the $i-$th electron.

(b) [2.5 points] Consider the same system as in part (a), but with the two particles interacting via an exchange term

$$H_{\text{ex}} = A\,\vec{\sigma}_1 \cdot \vec{\sigma}_2, \tag{1}$$

where A is a positive constant, and $\vec{\sigma}_1$, $\vec{\sigma}_2$ are the dimensionless spin operators of particle 1 and 2, respectively. Find the exact (i.e., not just in perturbation theory) five lowest energy eigenstates and eigenvalues. Assume that A is much smaller than the ground-state energy you found in part (a).

(c) [1.5 points] Now consider a **single** nonrelativistic spinless particle of mass m in an infinite square well potential of length L between $x = -\frac{L}{2}$ and $x = \frac{L}{2}$. The following time-dependent perturbation is added to the system:

$$H_1 = B \sin \omega t, \tag{2}$$

where B and ω are positive constants and t is the time. You may assume that B is very small and that H_1 has been applied for a long time. What are the selection rules for transitions generated by this perturbation (i.e., which eigenstates of the unperturbed system will be coupled by H_1)? Consider all possible values of ω.

(d) [3 points] Consider the same situation as in part (c), but now with

$$H_1 = Bx \sin \omega t, \tag{3}$$

where x is the position of the particle. What are the selection rules for transitions generated by this perturbation? Consider all possible values of ω.

PROBLEM 4: Stationary Perturbation Theory

Approximate the ammonia molecule NH_3 by a simple two-state system. The three hydrogen nuclei are in a plane and the nitrogen nucleus is at some fixed distance either above or below the plane of the hydrogen atoms. Each of the two states is approximately an eigenstate (a normalized up state $|e_1\rangle$ and a normalized down state $|e_2\rangle$) with the same energy E_0, where

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

However, there is a small amplitude A for transition from the up state to the down state. Thus the Hamiltonian of the two-state system can be written as $H = H_0 + H'$, where

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} ,$$

and

$$H' = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} .$$

Take A to be real with $|A| \ll |E_0|$.

(a) [3 points] Find the exact eigenvalues of H. What are the corresponding normalized eigenstates in terms of the up state $|e_1\rangle$ and the down state $|e_2\rangle$?

(b) [4 points] Next, suppose that the molecule is placed in an electric field that distinguishes the up and down states. The new Hamiltonian is $H = H_0 + H' + H''$ where

$$H'' = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}$$

with $\varepsilon_1 \neq \varepsilon_2$. What are the exact energy eigenvalues of the new Hamiltonian H?

(c) [3 points] When $|\varepsilon_1|, |\varepsilon_2| \ll |A|$, what answer does time-independent perturbation theory give for the energies of H to lowest non-vanishing order in ε_i? Compare your result to the exact answer.

PROBLEM 5: Spin Operators

To study physics of spin, we often define spin operators as

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and

$$S_\pm = \frac{1}{\sqrt{2}}(S_x \pm iS_y)$$

for spin$-1/2$ particles.

(a) [2 points] Express S_+ and S_- as 2×2 matrices and determine the commutation relation $[S_+, S_-]$.

(b) [2 points] Find the eigenvalues ($\lambda_1 \geq \lambda_2$) for the spin matrix operator S_y.

(c) [2 points] Find the normalized eigenvectors (u_1 and u_2) as column matrices for the matrix operator S_y.

(d) [2 points] Construct a unitary matrix U with the normalized eigenvectors in part (c). Then apply a similarity transformation to diagonalize S_y with the unitary matrix U.

(e) [2 points] If a particle is in the state with $s_z = -1/2$, and S_y is measured, what are the possible outcomes and their probabilities?