Quantum Mechanics
Qualifying Exam - Spring 2024

Notes and Instructions

- *There are five problems but only four problems will count to your grade.* If you chose to solve all five, the problem you score the least will be discarded. Attempt at least four problems as partial credit will be given.
- Write your alias on the top of every page of your solutions. *Do not write your name.*
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3).
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

One dimensional simple harmonic oscillator operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\,, \qquad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\mathbf{e}^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\
Y_2^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}(\sin\theta\cos\theta)\,\mathbf{e}^{\pm i\phi} \\
Y_2^{\pm2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathbf{e}^{\pm 2i\phi}
\end{aligned}$$

PROBLEM 1: Step potential

Consider a particle with $E > V_0 > V_1$ approaching a one-dimensional asymmetric step potential:

$$V(x) = \begin{cases} 0 & -\infty < x < 0 \\ V_0 & 0 \leq x \leq a \\ V_1 & a < x < \infty \end{cases}$$

a) Write the general form of the wave function in the three regions. (1 point)

b) What are the boundary conditions for the wave function at the points $x = 0$ and $x = a$? (2 points)

c) Using the fact that the probability current

$$J(x) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*(x)\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\right]$$

is uniform, derive the relationship between the transmission and reflection probabilities T and R. (3 points)

d) Compute the transmission probability T. (4 points)

PROBLEM 2: Postulates of quantum mechanics

Let us consider the following operators on a Hilbert space V^3:

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) Find the eigenvalues ℓ_z and normalized eigenvectors $|\ell_z\rangle$ of L_z. (1 point)

b) Take the state in which $\ell_z = -1$. In this state, what are the expectation values $\langle L_x\rangle, \langle L_x^2\rangle$, and the uncertainty ΔL_x? (3 points)

c) Find the eigenvalues and the normalized eigenvectors of L_x. (3 points)

d) If the particle is in the state with $\ell_z = 1$, and L_x is measured, what are the possible outcomes and their probabilities? (3 points)

PROBLEM 3: Perturbation theory

A particle of mass m moves non-relativistically in the three dimensional harmonic oscillator potential.

$$V = \frac{1}{2}k(x^2 + y^2 + z^2)$$

a) What is the energy and degeneracy of the ground state and first excited state. (1 point)

Now consider a small perturbation to the potential, so the potential is now

$$V = \frac{1}{2}k(x^2 + y^2 + z^2 + \lambda xy),$$

where λ is a small parameter.

b) What is the ground state energy to first order in perturbation theory. (2 points)

c) What is the ground state energy to second order in perturbation theory. (2 points)

d) Calculate the first excited energy levels to first order in perturbation theory. (3 points)

e) After the perturbation the first excited state splits into multiple energy levels. What are the wave functions for these new energy levels? (2 points)

PROBLEM 4: Addition of angular momenta

Two spin-$\frac{1}{2}$ particles have total spin operator $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, with S_z the corresponding z-component and $\mathbf{S}_i$ the spin operator of particle $i = 1, 2$.

a) What are the possible quantum numbers of S^2 and S_z? (1 point)

b) Find the normalized total spin states $|s, m\rangle$ in terms of $|s_1, m_1\rangle|s_2, m_2\rangle$ where m, m_1 and m_2 are the spin magnetic quantum numbers. It may or may not be helpful to use the spin raising/lower operator

$$S_\pm|s, m\rangle = \hbar\sqrt{(s \mp m)(s \pm m + 1)}|s, m \pm 1\rangle.$$

(3 points)

c) If S^2 is measured in the $|\frac{1}{2}, m_1 = \frac{1}{2}\rangle|\frac{1}{2}, m_2 = -\frac{1}{2}\rangle$ state, what are the possible outcomes and their probabilities? (1 point)

d) The same state of part c) is prepared at $t = 0$. Find the time evolution of this state under the Hamiltonian

$$\mathcal{H} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2.$$

and find the probability that the same state will be recovered at a later time t. (3 points)

e) Suppose you add a third spin-1 particle with spin $\mathbf{S}_3$. What are the possible quantum numbers of S^2 and S_z, where now $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$? (2 points)

PROBLEM 5: Coherent states

The ground state of the harmonic oscillator has the minimum possible product of uncertainties $\sigma_x \sigma_p$. There exists another kind of state which also minimizes the uncertainty, called a *coherent state*. In the harmonic oscillator, a coherent state $\psi_\alpha(x)$ is an eigenfunction of the lowering operator a with eigenvalue α,

$$a\psi_\alpha(x) = \alpha\, \psi_\alpha(x)\,. \tag{1}$$

Different coherent states have different values of α. Do *not* in general assume that the constant α is real.

a) Evaluate $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$ and $\langle p^2 \rangle$ for the coherent state wavefunction ψ_α of the harmonic oscillator in terms of α and constants. You may assume that $\psi_\alpha(x)$ is normalized. (2 points)

b) Using your result in part a), calculate σ_x and σ_p, and check whether a coherent state minimizes the Heisenberg uncertainty relation, that is, check whether $\sigma_x \sigma_p \geq \hbar/2$. (2 points)

c) Is the ground state wavefunction $u_0(x)$ of the harmonic oscillator a coherent state? If yes, for what value of α? (1 point)

d) Any wavefunction can be expressed as a linear combination of harmonic oscillator eigenfunctions $u_n(x)$. Assume therefore that the coherent state (1) can be written as

$$\psi_\alpha(x) = \sum_{n=0}^{\infty} c_n u_n(x)\,,$$

for some constants c_n (which may depend on the value of α). Using the properties of lowering operators a on $u_n(x)$, show that c_n are given by

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\, c_0\,.$$

(3 points)

e) Assume that $\Psi_\alpha(x, t = 0) = \psi_\alpha(x)$. Using your result in part d), show that $\Psi_\alpha(x, t)$ is still a coherent state — that is, show it satisfies

$$a\Psi_\alpha(x, t) = \alpha(t)\Psi_\alpha(x, t)\,.$$

What is $\alpha(t)$ in terms of α and other quantities? (2 points)