Quantum Mechanics
Qualifying Exam - January 2021

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write your alias on the top of every page of your solutions. Do not write your name.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3).
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

One dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta \mathbf{e}^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\
Y_2^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}(\sin\theta\cos\theta)\,\mathbf{e}^{\pm i\phi} \\
Y_2^{\pm2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathbf{e}^{\pm 2i\phi}
\end{aligned}$$

PROBLEM 1: One Dimensional Scattering

A 1D plane wave beam of particles with energy $E_0 > 0$ and mass m travels in the $+x$ direction and is described by a wavefunction $\psi(x)$. The potential $V(x) = 0$ for $x < 0$ but is non-zero for $x \geq 0$.

a) Derive the wavelength λ_0 of the particles for $x < 0$ (1 point).

b) Sketch $\text{Re}[\psi(x)]$ and $|\psi(x)|^2$ as a function of position when:

i) $V(x) = -E_0$ for $x > 0$ (2 points).

ii) $V(x) = E_0/2$ for $x > 0$ (2 points).

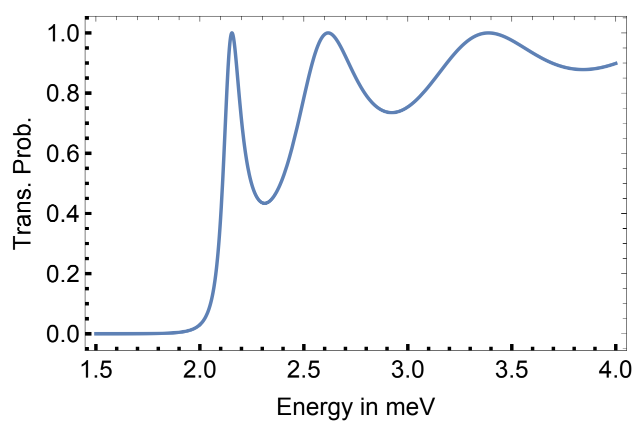
ii) $V(x) = E_0 x/(2\lambda_0)$ for $x > 0$ (2 points).

In all cases, your plot should range between $-2\lambda_0 \leq x \leq 2\lambda_0$.

c) Suppose the potential $V(x)$ is a square barrier with the form

$$V(x) = \begin{cases} V_0 & \text{for } 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}.$$

The transmission probability as a function of the energy of the particle E_0 is measured to be given by the figure below:



Use the information in the graph above to estimate the height of the barrier $V_0 > 0$ and its width a. Your physical reasoning is more important than getting an exact answer. (3 points)

Useful information:

$$\frac{\hbar^2}{2ma_0^2} = 13.6\text{eV}, \qquad \text{Bohr radius: } a_0 = 5.29 \times 10^{-11}\text{m}.$$

PROBLEM 2: Matrix Mechanics

A two-state quantum system is in the quantum state $|\psi\rangle$. Consider the operator

$$\hat{O}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The normalized eigenstates of $\widehat{O}_1$ are labelled $|A\rangle$ and $|B\rangle$, where $|A\rangle$ corresponds to the smaller eigenvalue.

a) The system is prepared in the state $|\psi\rangle$ given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}|A\rangle + \frac{e^{i\theta}}{\sqrt{2}}|B\rangle.$$

Determine the possible values obtained in the measurement of $\hat{O}_1$ and the probability of obtaining each value (1 point).

b) Consider the operator

$$\hat{O}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The normalized eigenvectors of $\hat{O}_2$ are $|C\rangle$ and $|D\rangle$ where $|C\rangle$ is the eigenvector corresponding to the smaller eigenvalue c and $|D\rangle$is the eigenvector corresponding to the larger eigenvalue d. The system is still in the state $|\psi\rangle$. The observable $\hat{O}_2$ is measured. It is found that the smaller eigenvalue is observed thrice as often as the larger eigenvalue. Determine θ. (3 points)

c) If the state $|C\rangle$ is *filtered out* from the state $|\psi\rangle$, what is the probability of measuring the smallest eigenvalue of $\hat{O}_1$? Regarding this experiment, suppose the possible outcomes of the measurement of $\hat{O}_1$ are spatially well separated in the detector. If the intensity of the signal is equal to the square of the local amplitude of the wavefunction at the detector, what would be the relative intensity of the measured signal compared to $\langle\psi|\psi\rangle$? (3 points)

d) Go back to the beginning of the problem and assume that the system is in a state $|\phi\rangle = e^{-i\theta/2}|\psi\rangle$. The operator $\hat{O}_2$ is measured. Determine the uncertainty of the measurement as a function of θ. (3 points)

PROBLEM 3: Hydrogen Atom

Imagine a hydrogen atom having a wavefunctinon at $t = 0$ that is a superposition of energy eigenstates, $\psi_{n\ell m}(r,\theta,\phi) = \langle r,\theta,\phi|\psi_{n,\ell,m}\rangle$, such that

$$|\psi(t=0)\rangle = A|\psi_{100}\rangle + B|\psi_{210}\rangle.$$

Here, $\psi_{100}(r,\theta,\phi) = R_{10}(r)Y_0^0(\theta,\phi)$ and $\psi_{210}(r,\theta,\phi) = R_{21}(r)Y_1^0(\theta,\phi)$, where $Y_\ell^m(\theta,\phi)$ are spherical harmonics. $R_{10}(r)$ and $R_{21}(r)$ are real. For this problem, you can express your answers in terms of $R_{n\ell}(r)$ functions.

a) If $n = 1$ has energy E_1 and $n = 2$ has energy E_2 for any ℓ and m, what is the expectation value, $\langle E\rangle$, of the electron energy at $t = 0$? Express your answer in terms of A. (2 points)

b) Find the time dependence of $\psi(t)$ for $t \neq 0$. How does the probability of measuring the $|\psi\rangle(t)$ state in the ground state depend on time? (2 points)

c) If we let:

$$\frac{E_2 - E_1}{\hbar} \equiv \omega_{12},$$

find the probability density distribution $|\psi(\mathbf{r},t)|^2$ of the electron in terms of r, θ, ϕ and ω_{12}. Assume A and B are real parameters. (3 points)

d) If the electron charge distribution is given by $-e|\psi(\mathbf{r},t)|^2$, does the state $|\psi(t)\rangle$ of the hydrogen atom exhibit a nonzero dipole moment at $t = 0$ or any other time? Explain. (3 points)

PROBLEM 4: Identical Particles

Two non-interacting electrons are placed in 1D infinite well with single-particle energy eigenstates E_n. Denote the nth (normalized) single-particle energy eigenstate as $|n\rangle$, and the spin part of the wave function as $|+\rangle$ or $|-\rangle$, *i.e.* a state in which the first electron is spin-up in the $n = 3$ single-particle state and the second electron is spin-down in the $n = 1$ single-particle state should be denoted: $|3\rangle|1\rangle \otimes |+\rangle|-\rangle$.

a) What are the possible wave functions which correspond to the ground state of the system? (2 points)

b) In terms of the single-particle ground state energy, what is the energy of the ground state of the system? (1 point)

c) What are the possible wave functions which correspond to the 1st excited state of the system? (2 points)

d) In terms of the single-particle ground state energy, what is the energy of the 1st excited state of the system? (1 point)

e) The electrons are both prepared to be spin up, in the lowest energy state that permits this. If their positions are measured, what is the probability that both electrons are found in the left half of the well? (4 points)

PROBLEM 5: Variational Method

The Hamiltonian of a one dimensional harmonic oscillator is

$$\mathcal{H} = T + V = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

where P and X are momentum and position operators respectively.

Let us employ a variational method with the following normalized trial wave function as the first excited state wave function

$$\langle x|\psi\rangle = \psi(x) = \sqrt{2\beta^3}x\mathrm{e}^{-\beta|x|},$$

where $\beta > 0$ is a real parameter.

a) Find the expectation value of the kinetic energy $\langle\psi|T|\psi\rangle$ in the trial wave function state (3 points)

b) Calculate the expectation value of the potential energy $\langle\psi|V|\psi\rangle$ in the same state. (2 points).

c) Plot $\langle\psi|(T+V)|\psi\rangle$ as a function of β. Determine the value of β that minimizes $\langle\mathcal{H}\rangle$ and find the first excited state energy E_1 with the variational method, such that $E_1 = \langle\mathcal{H}\rangle_{\mathrm{min}}$. (3 points)

d) How does E_1 compare with the *exact* energy of the first excited state? What feature in the wavefunction would allow you to distinguish the first excited state (exact or approximate) from the ground state? (2 point)

Problem 6: Spin 1 particle

A particle with spin $s = 1$ is under an external magnetic field along the x direction, $\mathbf{B} = B\hat{x}$. The corresponding Zeeman Hamiltonian is

$$\mathcal{H} = g\mathbf{B} \cdot \mathbf{S}.$$

a) Derive the spin matrices S_x, S_y and S_z in the basis of the $\mathbf{S}^2$ and S_z eigenstates, $|s, m_s\rangle$. (3 points)

b) If the particle is initially prepared to be in the $|1, 1\rangle$ state ($t = 0$), find the evolved state at $t > 0$. (3 points)

c) Using your result in b), what is the probability of finding the particle in the $|1, -1\rangle$ state? (2 points)

d) Calculate the equations of motion of the spin operator $\mathbf{S}(t)$ in the Heisenberg picture and solve them. Express your solution in terms of $\mathbf{S}(0)$ and interpret your result. (2 points)

Raising and lowering spin operators:

$$S_{\pm}|s, m_s\rangle = \hbar\sqrt{(s \mp m_s)(s \pm m_s + 1)}|s, m_s \pm 1\rangle.$$

$$S_{\pm} = S_x \pm iS_y.$$