Quantum Mechanics
Qualifying Exam - Spring 2020

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{2}$$

Harmonic oscillator wave functions

$u_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$

$u_1(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega x^2}{2\hbar}}$

Problem 1: Quantum Mechanical Particle (10 pts)

Consider a particle with mass m and charge e, in a magnetic field $\vec{B} = \nabla \times \vec{A}$. The Hamiltonian of the particle is given by

$$H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A}(\vec{r}))^2 \ , \qquad (3)$$

where $\vec{p}$ denotes the momentum of the particle and $\vec{r}$ denotes its position. Take $\vec{A} = -B\,y\hat{x}$, where B is the magnetic field strength and $\hat{x}$ denotes the unit vector along the x-direction.

Now let us do quantum mechanics with this particle.

(a) [3 pt] Show that the quantum mechanical operators $\hat{p}_x$ and $\hat{p}_z$ are constants of the motion.

(b) [3 pt] Suppose you were to write down the energy eigenfunction of the particle in the following way:

$$|\psi(x, y, z)\rangle \ = \ f(x, p_x, z, p_z)\, |\phi(y)\rangle \qquad (4)$$

where f is a function that contains the entire x and z dependence of the state of the particle while p_x and p_z are the constant values of the operators $\hat{p}_x$ and $\hat{p}_z$, respectively. Using part (a), find the function f.

(c) [4 pt] Derive the eigenvalue equation satisfied by $|\phi(y)\rangle$ and obtain the energy eigenvalues of the quantum mechanical particle.

Problem 2: Perturbation Theory (10 pts)

Consider a particle of mass m in a 1D infinite square well of width a:

$$V(x) = 0, \quad 0 \le x \le a \qquad V(x) = \infty, \quad x < 0, x > a. \tag{5}$$

a) [1 pt] Derive the energy eigenvalues and eigenstates for the particle. Show how you calculated your results. Be sure to normalize your states.

You might find useful the integral:

$$\int_0^{n\pi} \sin^2 y \; dy = \frac{n\pi}{2} \tag{6}$$

b) [1 pt] The quantum well is perturbed by an external potential of the form:

$$V_1(x) = \epsilon \frac{x}{a} \tag{7}$$

where ϵ is a positive energy.

Determine the shift in energy of the n^{th} eigenstate of the unperturbed well, $\psi_n(x)$, to first order in ϵ. You may not have to do any complicated calculations, but you need to justify your answer.

c) [2 pts] Next, instead of the perturbation from part b, let the perturbation be of the form:

$$V_2(x) = \epsilon \; a \; \delta\left(x - \frac{a}{2}\right) \tag{8}$$

where ϵ is a constant energy and $\delta(x)$ is the usual Dirac delta function.

Determine the shift in energy of the n^{th} eigenstate of the unperturbed well, $\psi_n(x)$, to first order in ϵ.

d) [3 pts] Using the perturbation $V_2(x)$, solve for an expression for the second order energy shift, in ϵ, of the n^{th} eigenstate of the unperturbed well, $\psi_n(x)$. Your answer should be given as a sum, but you should simplify your result as much as possible.

e) [3 pts] Using the results from part d, determine the magnitude of the second-order energy shifts to the $n = 1$ and $n = 2$ unperturbed energy eigenstates. If you can't determine an exact answer, give an approximation to the energy shifts. Your answers should be in terms of m, a, ϵ, and constants.

Note that if you give an approximation to either of the energy shifts you should justify the approximations that you make.

Problem 3: Identical particles (10 pts)

Three particles can exist each one in any of the three distinct 1-particle states $|n\rangle$, with $n = 1, 2, 3$. We define $\psi_n(x_i) \equiv \langle x_i|n\rangle$ as the wavefunction of particle $i = 1, 2, 3$ in the n state. Assume that the particles need *not* be in different states.

a) [2 pts] If the particles are distinguishable, how many different three-particle states are available? Justify your answer.

b) [4 pts] If they are identical bosons, write down all possible three-particle state wavefunctions.

c) [2 pts] Now assume that they are identical fermions. Write down all possible three-particle state wavefunctions.

d) [2 pts] Suppose that three electrons occupy a $2p$ shell of an atom ($\ell = 1$), each one having the *same* spin. What is the *total* magnetic quantum number of the allowed three-particle states? Justify.

Problem 4: Delta function and Atomic Model (10 pts)

Consider a delta-function potential $V(x) = -V_0\delta(x)$.

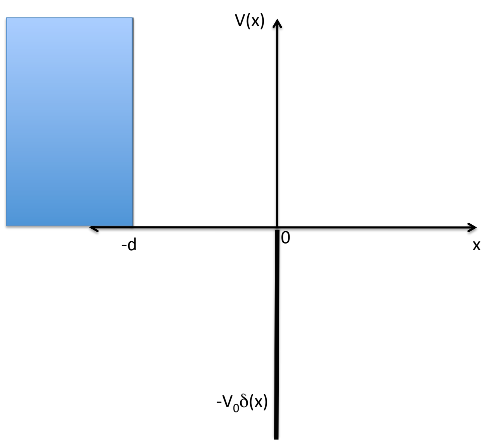
a) [1 pts] Calculate the bound state energy for this potential

b) [2 pts] Calculate the normalized bound state wave function and provide a sketch of the wave function.

Now consider a different potential. An approximate model for an atom near a wall is to consider a particle moving under the influence of the one-dimensional potential given by

$$V(x) = \begin{cases} -V_0\delta(x), & \text{x > -d} \\ \infty, & \text{x < -d} \end{cases}$$

As shown below



c) [3 pts] Find the transcendental equation for the bound state energies

d) [3 pts] What is the exact condition on V_0 and d for the existence of at least one bound state. (Sketching the functions on either side of the equal sign of your transcendental equation is useful.)

e) [1 pt] Show that when the delta function is far away from the wall, the bound state energy matches your answer to part a).

Problem 5: Simple Harmonic Oscillator (10 pts)

Possibly useful integral:

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} = \sqrt{\frac{\pi}{a}} e^{\left(\frac{b^2-4ac}{4a}\right)} \tag{9}$$

1) [5 pts] Consider a simple harmonic oscillator in one dimension with Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{10}$$

a) [5 pts] The ground state wave function can be expressed as $\psi_0(x) = Ae^{-Bx^2}$. Derive the constants A and B, as well as the energy associated with this state.

b) [3 pts] At time $t = 0$, an electric field with magnitude E is turned on, perturbing the Hamiltonian with a term given by

$$\hat{H}' = qE\hat{x} \tag{11}$$

Define a new operator $\hat{x}' \equiv \hat{x} + \frac{qE}{m\omega^2}$. Calculate $\hat{x}'^2$, and express the Hamiltonian in terms of this result. What is the new ground state wavefunction and energy?

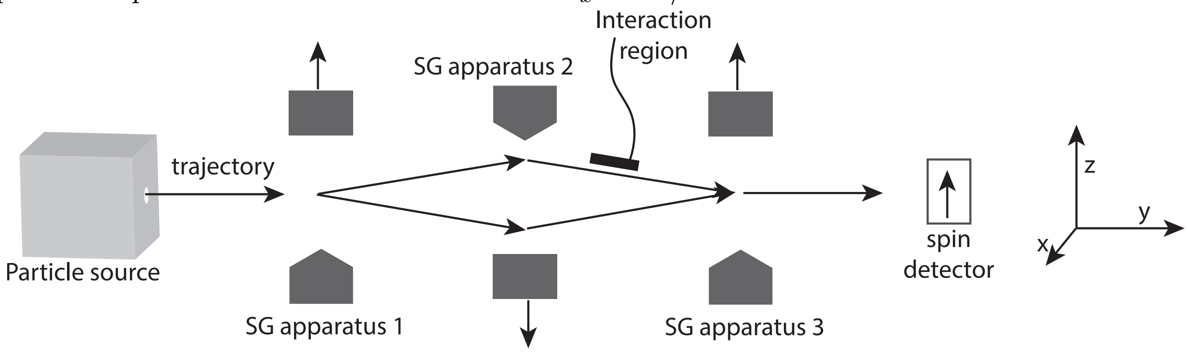
c) [2 pts] Assuming the oscillator was in the ground state for $t < 0$, what is the probability for the particle to be found in the grade state at $t = 0$?

Problem 6: Spin-1/2 Interferometer (10 pts)

Consider the experiment depicted in the figure below. All directions are given with respect to the axes shown on the right hand side.

On the left is a source that can emit single spin-1/2 particles, one at a time, each in the state $s_x = +1/2$, and each with the same linear momentum. The particles pass one by one through a sequence of Stern-Gerlach magnets that form an interferometer. Each Stern-Gerlach magnet is oriented with its field gradient parallel or anti-parallel to $\hat{z}$.

There is an interaction region on the upper path of the interferometer, as shown. Initially the interaction region is empty and when the spin detector on the far right measures each particle's spin in the x-direction it measures $s_x = +1/2$.



(a) [2 pts] We place equipment in the interaction region in the upper arm of the interferometer to measure each particle's spin along the z-axis. (Assume that the detector is perfect and does not miss any particles entering the upper arm of the interferometer). State what fraction of the particles leaving the source will be found in the upper arm of the interferometer, what values of s_z would be detected there, and the relative probability of each value of s_z.

(b) [1 pt] We remove the spin detector from the interaction region and replace it with a device that measures with certainty whether or not each particle traverses the upper arm of the interferometer, with no effect on the particle's spin or momentum. We do not measure its spin. In the case that a particle is observed to pass through the upper arm, what is the probability that the spin detector on the far right will measure an eigenvalue of $s_x = -1/2$?

(c) [1 pt] In the case that a particle is was *not* observed to pass through the upper arm, but arrived at the spin detector on the far right, what is the probability that the spin detector on the far right will measure an eigenvalue of $s_x = -1/2$?

(d) [6 pts] Now suppose that the interaction region instead rotates the spin about the $\hat{z}$ axis by an angle θ such that the spinor transform by the matrix $R_z(\theta)$

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

Derive the probability of measuring $s_x = 1/2$ at the spin detector.