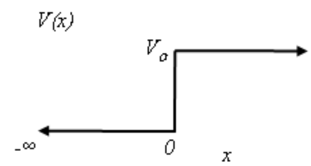
QUANTUM QUALIFIER EXAM, JANUARY 2007

PROBLEM 1



Consider the step potential shown in the figure.

a) [1 pts] Consider a particle traveling from $x = -\infty$ to the right with an energy E. The appropriate wavefunction for this particle is given by

$$\phi = \begin{cases} e^{ik_L x} + Ae^{-ik_L x} & \text{for } x < 0 \\ Be^{ik_R x} & \text{for } x > 0 \end{cases}$$

Give expressions for k_L and k_R and define any undefined parameters/constants given in your expression.

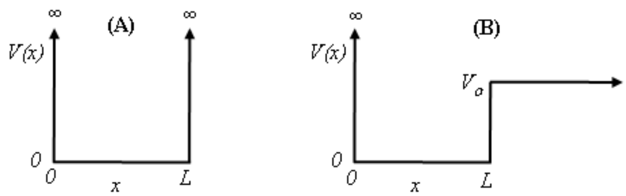
b) [3 pts] For the case that $E > V_o$, use appropriate boundary conditions to find the coefficients A and B.

c) [2 pts] For the case that $E > V_o$, find the probability that the particle will be reflected.

d) [2 pts] For the case that $E > V_o$, the probability that the particle will be transmitted is given by $T = 1 - R$. Determine and explain the physical meaning of the ratio $|B|^2/T$.

e) [2 pts] What is the probability for reflection when $E < V_o$?

PROBLEM 2



a) [2 pts] Calculate the energy eigenvalues for a particle of mass m in the one-dimensional infinite well shown in Figure A.

b) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), find a transcendental equation in E giving the eigenenergies in terms of V_o, L, m, and $\hbar$

c) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), what is the smallest value of V_o that gives one bound state? What is the smallest value of V_o that gives two bound states?

PROBLEM 3

Consider a quantum mechanical system that consists of two identical spin 1/2 particles that are fixed in space, separated by a distance d. Particle 1 is at the origin ($\vec{r}_1 = \vec{0}$) whereas particle 2 is at $\vec{r}_2 = d\ \hat{e}_z$. Each particle has a magnetic moment

$$\vec{\mu}(j) = \frac{g\mu_o}{\hbar}\vec{S}(j), \quad j = 1, 2$$

and a $g-$factor $g = 2$. $\vec{S}(j)$ is the spin operator of the j^{th} particle. Throughout this problem we will use the basis states $|1\rangle = |+,+\rangle$, $|2\rangle = |+,-\rangle$,$|3\rangle = |-,+\rangle$, and $|4\rangle = |-,-\rangle$, where these are the usual states written in terms of the $z-$components of the particles' spins.

a) [2pts] First consider what happens if we place the particles in an external magnetic field $\vec{B} = B\hat{e}_z$. Write the matrix representation for the Hamiltonian of the system $H_o = -\vec{\mu}\cdot\vec{B}$ in the $|i\rangle\,, i = 1, 2, 3, 4$ basis given above, considering only the interaction between the spins and the magnetic field. What are the energy eigenstates and eigenvalues for the system? Draw an energy-level diagram.

b) [3pts] We know, however, that the magnetic moment of each particle will create a magnetic field that the other particle will feel. The dipole field from particle 1 at particle 2 is (classically)

$$\vec{B}_{21} = \frac{1}{d^3}(3\mu_z(1)\hat{e}_z - \vec{\mu}(1))$$

so that the interaction Hamiltonian between the two particles is

$$\begin{aligned}\hat{H}' &= -\vec{\mu}_2 \cdot \vec{B}_{21} \\ &= \frac{g^2\mu_o^2}{\hbar^2 d^3}\left(-3S_z(1)S_z(2) + \vec{S}(1)\cdot\vec{S}(2)\right).\end{aligned}$$

Compute the action of the interaction Hamiltonian on each of the basis states. In other words, calculate $\hat{H}'|i>$ for $i = 1, 2, 3, 4$.
Hint: Use the usual angular momentum raising and lowering operators

$$\hat{S}^{\pm} = \hat{S}_x(j) \pm i\hat{S}_y(j), \; j = 1, 2$$

c) [2pts] Write the total Hamiltonian, $\hat{H} = \hat{H}_o + \hat{H}'$ as a matrix in the $|i\rangle$ basis.

d) [3pts] Find the eigenstates and eigenvalues of this total Hamiltonian and draw the energy level diagram as a function of the magnetic field strength.

PROBLEM 4

Consider a two state system described by the time-dependent Hamiltonian

$$H = \begin{pmatrix} 0 & \frac{\beta}{2}e^{i\omega t} \\ \frac{\beta}{2}^* e^{-i\omega t} & \hbar\omega_1 \end{pmatrix}$$

with

$$\vec{v}(t) = \begin{pmatrix} v_o(t) \\ v_1(t) \end{pmatrix}.$$

This is the Hamiltonian of a spin 1/2 particle in a strong magnetic field in the $\hat{z}$ direction combined with a rotating magnetic field in the x-y plane and models many NMR experiments. To analyze this system, it is convenient to write $\vec{v}(t)$ in terms of the time dependent vector $\vec{s}(t) = \begin{pmatrix} s_o(t) \\ s_1(t) \end{pmatrix}$ so that

$$\vec{v}(t) = \begin{pmatrix} s_o(t) \\ s_1(t)e^{-i\omega t} \end{pmatrix}.$$

For the case that $\beta = 0$ and $\omega = \omega_1$ (no rotating magnetic field), $s_o(t)$ and $s_1(t)$ are constant. The time dependence of $s_o(t)$ and $s_1(t)$ allows us to determine the probability that the rotating magnetic field induces a spin flip.

a) [1pt] Show that for $\beta = 0$, $\vec{v}(t)$ statisfies the time-dependent Schrodinger equation

$$H\vec{v}(t) = i\hbar\frac{\partial\vec{v}(t)}{\partial t}.$$

when when $s_o(t)$ and $s_1(t)$ are constant and $\omega = \omega_1$.

b) [3pts] For the case that β is a nonzero constant, use Schrödinger's equation for $\vec{v}(t)$ to show that $\vec{s}(t)$ evolves according to the effective Hamiltonian H' with

$$H' = \begin{pmatrix} 0 & \frac{\beta}{2} \\ \frac{\beta}{2}^* & \hbar\Delta\omega \end{pmatrix}$$

and

$$\Delta\omega = \omega_1 - \omega.$$

c) [3pts]Assuming the system starts in the state $\vec{s}(t) = \binom{0}{1}$ at $t = 0$, find $\vec{s}(t)$.

d) [3pts]Assuming the system starts in the state $\vec{s}(t) = \binom{0}{1}$ at $t = 0$, find the probability of finding the system in the state $\binom{1}{0}$ as a function of time.

PROBLEM 5

A particle of mass m is confined to a two-dimensional plane. The potential energy of the particle is

$$V(\rho) = \begin{cases} 0 & \rho < \rho_o \\ \infty & \rho \geq \rho_o, \end{cases}$$

where ρ is the radial coordinate of plane polar coordinate (ρ, φ). This potential confines the particle to the region of space $\rho \leq \rho_o$. The particle in this "circular square well" is the quantum analog of a marble on the head of a drum. The stationary-state Hamiltonian eigenfunctions of the particle are $\Psi_{n,m_\ell}(\rho, \varphi)$ with eigenenergies E.

a) [4pts] Write down a second-order differential equation for the radial function $R_{n,m_\ell}(\rho)$ in the bound-state Hamiltonian eigen functions

$$\psi_{n,m_\ell}(\rho, \varphi) = R_{n,m_\ell}(\rho)\Phi_{m_\ell}(\varphi),$$

where $\Phi_{m_\ell}(\varphi)$ is an eigenfunction of the orbital angular momentum operator $\hat{L} = -i\hbar\partial/\partial\varphi$. Write down and justify the boundary conditions that physically admissible solutions to your differential equation must satisfy, and write down the normalization integral for the radial functions.

b) [2pts] What, if anything, can you conclude from your differential equation about the degree of degeneracy of the bound-state energies E_{n,m_ℓ}. Justify your answer.

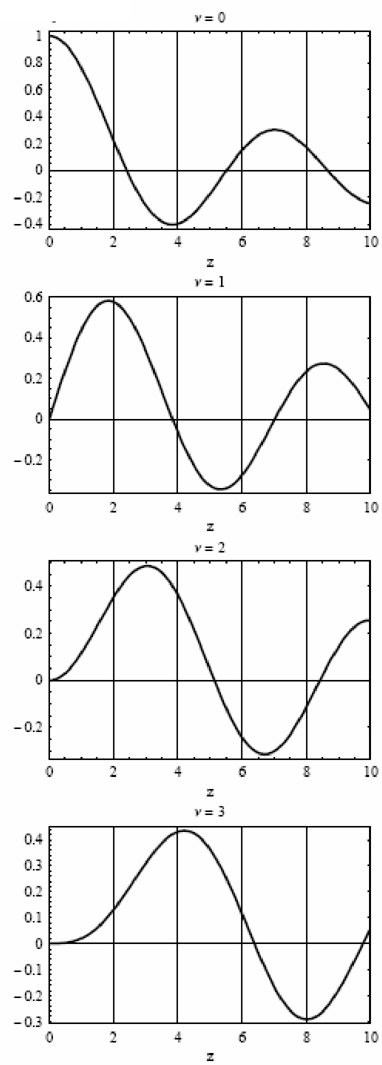
c) [2pts] Derive an equation for the bound-state energies E_{n,m_ℓ} in terms of the zeros $\varsigma_{n,\nu}$ of the cylindrical Bessel function of the first kind, $J_{\pm\nu}(z)$. (See the hint below.)

d) [2pts] Estimate the energies of the *lowest three* bound states of the cylindrical square well. Express your answers in terms of fundamental constants, the mass m, and the well radius ρ_o.

Hint: The cylindrical Bessel functions are solutions of Bessel's equation

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 - \nu^2)\right] J_{\pm\nu}(z) = 0$$

The so-called cylindrical Bessel functions of the first kind, $J_{\pm\nu}(z)$, are regular at the origin and normalizable. These functions oscillate with increasing z and have an infinite number of *nodes*, i.e., values for which $z = \varsigma_{n,\nu} > 0$ at which $J_{\pm\nu}(z) = 0$; these nodes are indexed by $n = 1, 2, ...$ The figure shows the first four cylindrical Bessel functions.



First four cylindrical Bessel functions of the first kind (for use in problem 5.)

PROBLEM 6

Consider an ensemble of identical particles whose state space is spanned by the basis

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assume that the Hamiltonian H and an observable A are represented by

$$H = \hbar\omega_o \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of H are $\hbar\omega$, $2\hbar\omega$, and $-\hbar\omega$ with eigenvectors given by

$$|\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad |2\hbar\omega\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } |-\hbar\omega\rangle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are $-1, 1,$and 1 with eigenvectors given by

$$|a_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, |a_{1,1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, |a_{1,2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For all times $t < 0$, the particles are in a state given by

$$|\psi_o\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix} .$$

a) [1pt] Write down an expression for the time evolution operator $U(t, t_o = 0)$ in Dirac notation

b) [2 pt] Determine $|\psi(t)\rangle$, the state vector at an arbitrary time.

c) [2 pt] What is the probability that a measurement of A at a time $t = 0$ yeilds $a = -1$?

d) [2 pt] What is the probability that a measurement of A at an arbitrary time t yields a value $a = -1$?

e) [3 pt] Assume that at $t = 0$ the operator A is observed to be 1. What is the probability that a short time later $(0 < t << 1/\omega)$, the eigenenergy of the system is observed to be $-\hbar\omega$?