Quantum Mechanics
Qualifying Exam–January 2006

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work**.

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_2^2(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{2i\varphi}$$

$$Y_2^1(\theta,\varphi) = -\frac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{i\varphi}$$

$$Y_1^1(\theta,\varphi) = -\frac{3}{\sqrt{8\pi}}\sin\theta\, e^{i\varphi} \qquad Y_2^0(\theta,\varphi) = \frac{5}{\sqrt{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

$$Y_1^0(\theta,\varphi) = \frac{3}{\sqrt{4\pi}}\cos\theta \qquad Y_2^{-1}(\theta,\varphi) = \frac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{-i\varphi}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{3}{\sqrt{8\pi}}\sin\theta\, e^{-i\varphi} \qquad Y_2^{-2}(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{-2i\varphi}$$

Angular momentum raising and lowering operators:

$$\hat{L}_\pm = (\hat{L}_x \pm i\,\hat{L}_y)$$

PROBLEM 1: Eigenvalues and Eigenvectors

Suppose the Hamiltonian for a system is given by

$$H = \hbar\omega_0(\sigma_x + \sigma_y)$$

where σ_x and σ_y are two of the Pauli matrices.

(a). Calculate the eigenvalues and eigenvectors for this Hamiltonian. (*5 points*)

(b). In the Schrödinger picture, the state vector is

$$|\psi(t)\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

At $t = 0$, the state of this system has $\alpha(0) = 0$ and $\beta(0) = 1$. Evaluate $\alpha(t)$ and $\beta(t)$ for $t > 0$. (*5 points*)

PROBLEM 2: One Dimensional Barrier

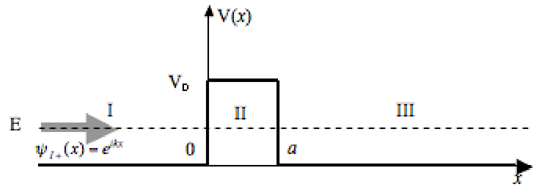

Figure 1: A one-dimensional rectangular barrier.

A stream of particles with kinetic energy $0 < E < V_0$ and mass m, traveling in the positive x direction is incident on a barrier from the left. These incident particles have a wave function

$$\psi_{\text{in}}(x) = e^{ikx}$$

where $k = \sqrt{2mE}/\hbar$.

(a). Write down the wave function in each region, ($\psi_{\text{I}}(x)$, $\psi_{\text{II}}(x)$, and $\psi_{\text{III}}(x)$), in terms of the eigenfunctions in each region, with undetermined coefficients. (*2 points*)

(b). State the boundary conditions at $x = 0$, $x = a$, and $x = \infty$. (*1 point*)

(c). Sketch the wavefunction in a single figure, showing its form in all regions. (*1 point*)

(d). Find the transmission coefficient (T) for particles with

$$\begin{aligned} k &= \sqrt{2mE}/\hbar \\ \lambda &= \sqrt{2m(V_0 - E)}/\hbar. \end{aligned}$$

(*4 points*)

(e). If the barrier of transmission is small ($\lambda a \ll 1$), show that

$$T \approx \frac{16E}{V_0}\left(1 - \frac{E}{V_0}\right) exp\left(-\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right) .$$

(*2 points*)

PROBLEM 3: Harmonic Oscillator in 3 Dimensions

Consider a particle subject to a 3-dimensional harmonic oscillator potential:

$$\begin{aligned} H &= H_x + H_y + H_z \\ &= \frac{P_x^2}{2m} + \frac{1}{2}m\omega_x^2 X^2 + \frac{P_y^2}{2m} + \frac{1}{2}m\omega_y^2 Y^2 + \frac{P_z^2}{2m} + \frac{1}{2}m\omega_z^2 Z^2 \end{aligned}$$

where $\omega_x = \omega_y = \omega$ and $\omega_z = 2\omega$. The wave function is given by:

$$\Psi_{n_x,n_y,n_z}(x,y,z) = \Phi_{n_x}(x)\Phi_{n_y}(y)\Phi_{n_z}(z)$$

(a). For the ground state wave function, determine, ΔX, ΔP_x and their product $\Delta X \Delta P_x$ using Dirac notation and raising and lowering operators. (*3 points*)

(b). For the ground state, $\Psi_{0,0,0}$, do you expect
(i) ΔZ to be larger or smaller than ΔX?
(ii) $\Delta Z \Delta P_z$ to be larger or smaller than $\Delta X \Delta P_x$?
Explain your reasoning in each case. (*2 points*)

(c). Assume that at $t = 0$ the particle is in the state:

$$|\Psi(t=0)> = \frac{1}{\sqrt{6}}|\Psi_{1,0,0}> + \frac{1}{\sqrt{3}}|\Psi_{2,1,0}> + \frac{1}{2}|\Psi_{0,1,0}> + \frac{1}{2}|\Psi_{1,0,1}>$$

If one measures the total energy, E, what is the probability of obtaining $5\hbar\omega$? (*2 points*)

(d). Immediately after the measurement performed in part (c), what harmonic oscillator state (or superposition of states) is the system in? (*1 points*)

(e). Assume that at time $t = 0$ the measurement described in part (c) is performed and that the energy is found to be $E = 5\hbar\omega$. If the observable, X, is measured at a time $t > 0$, will its probability distribution be be time dependent or time independent? Explain your reasoning. (*2 points*)

PROBLEM 4: Angular Momentum

Consider the problem of a 3D rigid rotator. Let the rotator be far removed from any forces so its energy is purely kinetic.

$$E = \frac{L^2}{2I}$$

where I is the moment of inertia. The quantum mechanical Hamiltonian operator is

$$H = \frac{\hat{L}^2}{2I}$$

(a). What are the frequencies of photons emitted due to the energy decay between successive levels of the rotator? (*3 points*)

At a given instant, the rigid rotator is in the state

$$\Psi(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin\theta \sin\phi$$

(b). What possible values of L_z will be found in a measurement, and with what probabilities? (*3 points*)

(c). What is $\langle L_x \rangle$ for this state? (*2 points*)

(d). What is $\langle L^2 \rangle$ for this state? (*2 points*)

PROBLEM 5: Variational Method

Let us consider an electron moving in a Coulomb potential:

$$H = T + V = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}$$

where m is the mass of the electron.

Let us choose

$$\psi_\alpha(r) = e^{-\alpha r^2}, \quad \alpha > 0$$

as a trial wave function for the ground state.

(a). Find $\langle \psi_\alpha | \psi_\alpha \rangle$. (*1 points*)

(b). Find the expectation value of the Hamiltonian $\langle H \rangle$. (*3 points*)

(c). Determine a bound on the ground state energy of this system using the variational method and this trial wavefunction. Express your answer in terms of

$$\mathrm{Ry} \;=\; \frac{me^4}{2\hbar^2}.$$

(*5 points*)

(d). How does this compare to the actual ground state energy of the system? Be specific. (*1 point)*

PROBLEM 6: Perturbation Theory

Consider a particle confined to a ring of unit radius. You are told that the Hamiltonian operator can be written as $H = H_0 + H_1$ where

$$H_0 = (i\frac{\partial}{\partial\theta} + a)^2$$

and

$$H_1 = V_0 \cos\theta \ .$$

The parameter a is a tunable constant in this toy model. (In this problem, units are chosen such that $\hbar$ = 2m =1)

(a). Find the complete set of eigenvalues and eigenfunctions of H_0 (*2 points*)

(b). Use perturbation theory to find the first order correction to the ground state energy of H_0 due to the perturbation H' for $0 < a < 1/2$. (*2 points*)

(c). Use perturbation theory to find the second order corrections to the ground state energy of H_0 due to the perturbation H_1 for $0 < a < 1/2$. (*3 points*)

(d). For $a = 1/2$, the ground state energy of H_0 is degenerate. Find the first order correction to the energy for his case. (*3 points*)