Quantum Mechanics
Qualifying Exam – August 2022

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work**.

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p} \;=\; -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^\dagger), \quad \left[\hat{a},\hat{a}^\dagger\right] \;=\; 1, \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle, \quad \text{and} \quad \hat{a}^\dagger|n\rangle \;=\; \sqrt{n+1}|n+1\rangle\ . \end{aligned}$$

The Hermite polynomials:

$$\begin{aligned} H_0(y) &= 1, \quad H_1(y) \;=\; 2y, \quad H_2(y) \;=\; 4y^2-2 \\ H_n(y) &= (-1)^n e^{y^2}\frac{\partial^n}{\partial y^n}e^{-y^2} \end{aligned}$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}} \qquad Y_2^{\pm 2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{\pm 2i\varphi}$$

$$Y_1^{\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} \quad Y_2^{\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$$

Angular momentum raising and lowering operators:

$$\begin{aligned} L_\pm &= L_x \pm i\, L_y \\ L_+|\ell,m\rangle &= \hbar[\ell(\ell+1)-m(m+1)]^{1/2}|\ell,m+1\rangle \\ L_-|\ell,m\rangle &= \hbar[\ell(\ell+1)-m(m-1)]^{1/2}|\ell,m-1\rangle \end{aligned}$$

Gaussian Integral:

$$I_0(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = (\pi/\alpha)^{1/2}\ , \qquad \alpha > 0$$

where α is usually chosen to be real.

PROBLEM 1: Infinite Square Well

(a) Consider a one-dimensional infinite square well:

$$U(x) = \begin{cases} 0 & -L < x < L \\ +\infty & \text{otherwise} \end{cases}$$

(i) [2 points] Show that even and odd eigenfunctions are given by

$$\begin{aligned} \psi_{\text{even}}(x) &= \frac{1}{\sqrt{L}} \cos\left(\frac{(2n+1)\pi x}{2L}\right) \qquad n = 0, 1, 2, 3, \cdots \\ \psi_{\text{odd}}(x) &= \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3 \cdots \end{aligned}$$

(ii) [3 points] Calculate the position uncertainty for even and odd eigenstates.

(iii) [3 points] Assume the particle is in the ground state. Suddenly, the width of the well doubles ($-2L < x < 2L$). Immediately after the well doubles, what is the probability to find the particle in the (new) ground state?

(b) [2 points] Consider a one-dimensional infinite square well with an attractive ($\alpha > 0$) delta function at its center:

$$U(x) = \begin{cases} -\alpha\delta(x) & -L < x < L \\ +\infty & \text{otherwise} \end{cases}$$

Find the eigenfunction corresponding to the first excited state.

Possibly useful integrals:

$$\begin{aligned} \int_{-L}^{L} x^2 \cos^2\left(\frac{n\pi x}{2L}\right) dx &= \frac{L^3}{3}\left(1 + \frac{6(-1)^n}{n^2\pi^2}\right) \\ \int_{-L}^{L} x^2 \sin^2\left(\frac{n\pi x}{2L}\right) dx &= \frac{L^3}{3}\left(1 - \frac{6(-1)^n}{n^2\pi^2}\right) \end{aligned}$$

PROBLEM 2: Harmonic Oscillator

(a) [3 points] The Schrödinger equation for the quantum mechanical harmonic oscillator can be written as

$$\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\hbar\omega\psi = E\psi\,,$$

where $\hat{a}_\pm$ are the ladder operators given by

$$\hat{a}_\pm = \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm im\omega x\right)\,.$$

If ψ satisfies the Schrödinger equation with energy E, show that $\phi = \hat{a}_-\psi$ also satisfies the Schrödinger equation but with energy $E - \hbar\omega$.

(b) [4 points] The harmonic oscillator has a lowest energy state represented by ψ_0. Application of the ladder operator $\hat{a}_-$ to this state generates a wavefunction that does not exist such that we can write,

$$\hat{a}_-\psi_0 = 0.$$

Use this equation to derive $\psi_0(x)$, the wavefunction for the ground state. Do not bother to normalize it.

(c) [3 points] Use your solution for $\psi_0(x)$ and the Schrödinger equation to determine the energy of the ground state of this system.

PROBLEM 3: Angular momentum

A particle in a central potential has an orbital angular momentum $\ell = 2\hbar$ and a spin $s = 1\hbar$.

(a) [2 points] Find the energy levels associated with the spin-orbit interaction term of the form $\hat{H}_o = A\vec{L} \cdot \vec{S}$ where A is a constant.

(b) [2 points] Find the degeneracy for each energy level.

(c)-(f) Now consider an electron in a state described by the wave function

$$\psi = \frac{1}{\sqrt{4\pi}}(e^{i\phi} \sin\theta + \cos\theta)g(r)\,,$$

where

$$\int_0^\infty |g(r)|^2 r^2 dr = 1\,,$$

where ϕ and θ are the azimuthal and polar angles respectively.

(c) [1 point] Show $\int |\psi|^2\, d^3x = 1$.

(d) [2 points] What are the possible results of a measurement of the z-component L_z of the angular momentum of the electron in this state?

(e) [2 points] What is the probability of obtaining each of the possible results in part (d)?

(f) [1 point] What is the expectation value of L_z?

PROBLEM 4: Spin

Consider the properties of a spin-1/2 particle. The spin physics is described by a two-dimensional space and the spin operators:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defined using the usual basis states

$$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle \tag{1}$$

The square of the "total" spin operator is

$$S^2 = S_x^2 + S_y^2 + S_z^2 \tag{2}$$

(a) [1 point] Show that S_x and S_z do not have simultaneous eigenvectors. Show that the eigenvectors of S_z are also eigenvectors of S^2. What are the eigenvalues? (Show your work)

(b) [1 point] For any operator $\hat{O}$ and state $|\chi\rangle$, define the (squared) uncertainty as:

$$\Delta^2\hat{O} = \langle\chi|\hat{O}^2|\chi\rangle - \langle\chi|\hat{O}|\chi\rangle^2 \tag{3}$$

For the state $|+\rangle$, what is the expectation value $\langle S_x\rangle$ and the uncertainty ΔS_x? Show your work and give a brief physical explanation of this result.

(c)-(g) Consider a particle initially (t=0) in the state

$$\chi = A\begin{pmatrix} 1+i \\ \sqrt{2} \end{pmatrix},$$

where A is a real constant.

The spin is in a magnetic field giving an interaction:

$$\hat{H} = -\mu B_0 S_z\,, \quad \hat{H}|\pm\rangle = \pm\hbar\omega_0|\pm\rangle \tag{4}$$

where $\omega_0 = \mu B_0/2$ will help simplify the notation.

(c) [2 points] What is the time-dependent expectation value of S_z?

(d) [1 point] For the situation described in Part (c), what are possible outcomes of a measurement of S_z and their probabilities as a function of time?

(e) [2 points] What are the eigenvalues and eigenvectors of S_x? Show your work.

(f) [2 points] Again if the particle is initially (t=0) in the state $|\chi\rangle$, what is the time-dependent expectation value of S_x?

(g) [1 point] For the situation described in Part (f), what are possible outcomes of a measurement of S_x and their probabilities as a function of time?

PROBLEM 5: Multi-fermion Systems

Consider two spin-half particles, both confined in an infinite potential well stretching from $-L/2$ to $L/2$. The particles do not interact with each other. Denote by $|n\rangle_i$ (with $i = 1, 2$) the energy eigenstate with level n of the ith particle. The single particle normalized energy eigenstates can be solved using the boundary conditions provided, resulting in standard expressions: $\langle x|n\rangle \sim cos(n\pi x/L)$ for n odd; $\langle x|n\rangle \sim sin(n\pi x/L)$ for n even. Further, denote by $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ (with $i = 1, 2$) the spin eigenstate of the ith particle.

Define the set $S = \{|n\rangle_i\,, |\uparrow\rangle_i\,, |\downarrow\rangle_i\}$ for all n, $i = 1, 2$ on which a tensor product $\otimes$ is naturally defined.

(a) [1 point] First warm-up question: construct symmetric and antisymmetric combinations of the spin basis vectors $|\uparrow\rangle_1, |\downarrow\rangle_1, |\uparrow\rangle_2$ and $|\downarrow\rangle_2$. Here, symmetry refers to exchange between particles 1 and 2.

(b) [1 point] Second warm-up question: construct symmetric and antisymmetric combinations of the spatial energy eigenstate basis vectors $|1\rangle_1, |1\rangle_2, |2\rangle_1, |2\rangle_2$. Here, symmetry refers to exchange between particles 1 and 2.

(c) [2 points] Using elements of the set S (specifically $|1\rangle_1, |1\rangle_2, |\uparrow\rangle_1, |\downarrow\rangle_1, |\uparrow\rangle_2$ and $|\downarrow\rangle_2$) and tensor products between them, write down the ground state $|\psi_1\rangle$ of the two fermion system. State whether $|\psi_1\rangle$ is overall symmetric or antisymmetric.

(d) [2 points] Next, consider the first excited energy eigenstate of the two-fermion system. There are four degenerate states at this level. Using the elements of S, (specifically $|1\rangle_1, |1\rangle_2, |2\rangle_1, |2\rangle_2, |\uparrow\rangle_1, |\downarrow\rangle_1, |\uparrow\rangle_2$ and $|\downarrow\rangle_2$), construct these states.

(e) [0.5 points] Let's introduce an interaction Hamiltonian of the form

$$\hat{H}_{\text{int}} \;=\; A\,\vec{S}_1 \cdot \vec{S}_2 \tag{5}$$

where $\vec{S}_1$ and $\vec{S}_2$ are the dimensionless spin operators of the two fermions with $\hbar = 1$ for simplicity. Recast $\hat{H}_{\text{int}}$ in terms of $\vec{S}^2$, where $\vec{S} = \vec{S}_1 + \vec{S}_2$.

(f) [1.5 points] Check whether the symmetric and antisymmetric spin states in Part (a) are eigenfunctions of $\hat{H}_{\text{int}}$ and write down the corresponding eigenvalues.

(g) [2 points] Check whether the ground state and first excited states of the *non-interacting* two fermion system that you wrote down in Parts (c) and (d), remain energy eigenstates once $\hat{H}_{\text{int}}$ is turned on. If the non-interacting ground state and first excited states had energy E_1 and E_2, respectively, what are the energies of these states after $\hat{H}_{\text{int}}$ is turned on?

PROBLEM 6: A Central Potential

Consider a pseudo-particle with mass μ subject to a central potential $V(r)$ in two-dimensional space, where r^2 is equal to x^2+y^2. The kinetic energy operator in Cartesian coordinates (x, y) and polar coordinates (r, φ) reads

$$\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \tag{6}$$

and

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right], \tag{7}$$

respectively.

(a) [1.5 points] Using the ansatz

$$\psi(r,\varphi)=\exp(\imath m\varphi)R(r), \tag{8}$$

derive the radial Schrödinger equation; carefully explain all steps.
You should find:

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{m^2}{r^2}\right)+V(r)\right]R(r)=ER(r). \tag{9}$$

(b) [1 point] What are the allowed values of m? Carefully explain your answer.

(c) [1 point] Provide a physical interpretation of the quantity m.

(d) [2.5 points] Introduce a scaled radial wave function $u(r)$ such that the radial kinetic energy does not contain a first derivative with respect to r.
You should find that $u(r)$ fulfills the following equation:

$$\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}+\frac{\hbar^2\left(m^2-\frac{1}{4}\right)}{2\mu r^2}\right]u(r)=Eu(r). \tag{10}$$

(e) [2 points] Consider the $m=0$ case. Assuming a bound state exists, is the ground state energy of the pseudo-particle in 2D more or less strongly bound than that in 3D. Explain. Please provide a mathematical as well as a pictorial/physical explanation.

(f) [2 points] Considering $m=0$ and a potential $V(r)$ that goes to infinity as $r\to 0$ and that supports a bound state with energy $E_{b.st.}$, what are the $r\to 0$ and $r\to\infty$ behaviors of $u(r)$?