E & M Qualifier

January 2026

There are 5 problems. 4 of the 5 questions count as your grade on the exam. You may choose to answer only 4 questions. If you do answer all 5, the question with the lowest grade will be dropped and the remaining 4 questions will be used to grade the exam.

To ensure that your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. each problem is presented in SI units. If you choose to use a different unit system, e.g. Gaussian, make a clear note of your unit choice at the start of your solution.
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

Two infinite parallel plates, one with a surface charge σ_0 in the $x = 0$ plane, and the other with a surface charge $3\sigma_0$ in the $x = d$ plane, are placed near each other as shown in Figure (a) below.

(a) Calculate the electric field $\vec{\mathbf{E}}$ everywhere around these two plates. [1 point]

(b) Assuming the electric potential on the first plate (the one with surface charge σ_0 is V_0), calculate the electric potential everywhere. [1 point]

For the rest of the problem, half of the space between the two plates is filled with an isotropic linear dielectric with dielectric constant $\epsilon_1 = 2\epsilon_0$ while the other half is filled with an isotropic linear dielectric with dielectric constant $\epsilon_2 = 4\epsilon_0$, as shown in Figure (b) below, where ϵ_0 is the vacuum dielectric constant.

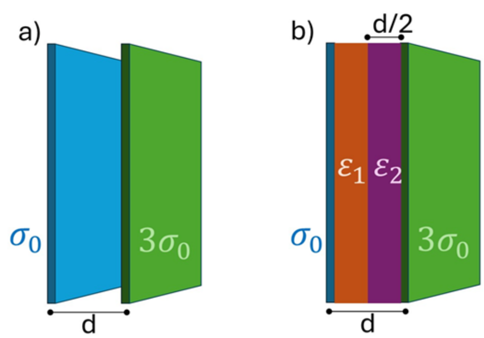
(c) Calculate the electric field $\vec{\mathbf{E}}$ everywhere. [2 points]

(d) Plot the electric field as a function of position. [1 point]

(e) Calculate the electric potential everywhere, again assuming that the electric potential of the leftmost plate is V_0. [2 points]

(f) Plot the electric potential as a function of position. [1 point]

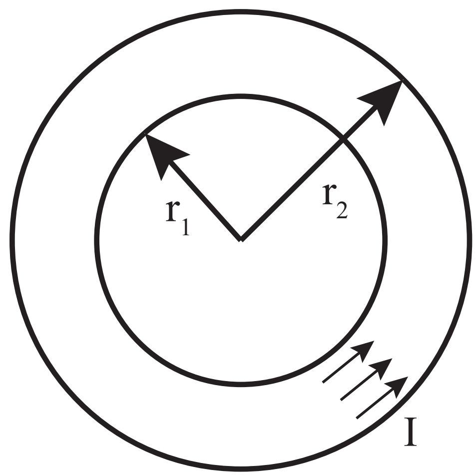
(g) Calculate the bound charge density everywhere. [2 points]



Problem 2: Magnetostatics

(a) Consider a magnetic field field $\vec{\mathbf{B}}$ that is uniform. Show that this situation is consistent with defining the magnetic vector potential in SI units to be $\vec{\mathbf{A}} = -\frac{1}{2}\left(\vec{\mathbf{r}} \times \vec{\mathbf{B}}\right)$, where $\mathbf{r}$ is the position vector of the point in question. [2 points]

(b) Calculate $\vec{\nabla} \cdot \vec{\mathbf{A}}$ in this case. What does your result mean? [1 point]



(c) Consider a thin metallic ribbon carrying a current I uniformly through its width. The ribbon is bent into the form of a circular ring of inner and outer radii r_1 and r_2, respectively, as shown in the figure above. Find the magnetic field $\vec{\mathbf{B}}$ at the center of the ring. What is its direction? [4 points]

(d) Find the magnetic moment $\vec{\mathbf{m}}$ associated with the ring. What is its direction? [3 points]

Problem 3: Plane Waves

In this problem we consider the propagation of an electromagnetic wave through an optically active medium; a medium that causes the direction of polarization to rotate about the direction of propagation. Such a medium can be described using the susceptibility tensor

$$\boldsymbol{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}.$$

Furthermore, recall that the polarization of the medium and the electric field are related by $\vec{P} = \epsilon_0 \boldsymbol{\chi} \vec{\mathbf{E}}$ in SI units. *You can assume a non-magnetic linear dielectric material with no free charges or currents.*

(a) Starting from the Maxwell equations, the expressions given in the statement of the problem, and assuming a transverse electromagnetic wave propagating in the z direction, derive the magnitude of the wave vector $\vec{\mathbf{k}}$ for the two possible polarizations in terms of the components of $\boldsymbol{\chi}$. [4 points]

(b) Show that the allowed wave modes correspond to circularly polarized waves. [3 points]

(c) Under the assumption that $\chi_{12} \ll \chi_{11}$, derive an expression for the amount that the polarization rotates over a distance ℓ as a function of χ_{11}, χ_{12}, and ω. The approximation $\sqrt{1 \pm \epsilon} \approx 1 \pm \epsilon/2$ might be useful. [3 points]

Problem 4: Gauges

Maxwell's Equations can be written in SI units as:

$$\vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_0 \tag{1}$$

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \tag{2}$$

$$\vec{\nabla} \times \vec{\mathbf{H}} = \vec{J}_0 + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{3}$$

$$\vec{\nabla} \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0 \tag{4}$$

(a) Explain the meaning of $\vec{\mathbf{D}}$ and $\vec{\mathbf{H}}$ and any restrictions on ρ_0 and $\vec{\mathbf{J}}_0$. Do not simply give formulas relating the fields, but explain their physical interpretation, and how they relate or do not relate to the force experienced by a moving test charge. [2 points]

(b) Show that the above equations allow us to introduce the scalar potential ϕ and the vector potential $\vec{\mathbf{A}}$ and give their relationship to the electric and magnetic fields. [2 points]

(c) Explain what it means to say that the electromagnetic fields are *gauge invariant*, and explain why this is important. [2 points]

(d) Consider a charge distribution $\rho(\vec{r}, t)$ and a current distribution $\vec{\mathbf{J}}(\vec{\mathbf{r}}, t)$ that exist in vacuum. In the *Lorenz gauge* the potentials satisfy

$$\vec{\nabla} \cdot \vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0.$$

Show that this condition allows you to write two uncoupled inhomogeneous wave equations for ϕ and $\vec{\mathbf{A}}$ in the vacuum, and write them out explicitly. [3 points]

(e) Is a gauge transformation the same thing as a Lorentz transformation for electromagnetic fields? If so, demonstrate why. If not, explain what they are and how they differ. [1 point]

Problem 5: Multipole Expansions

The Coulomb potential for a point charge Q located at source point $\vec{\mathbf{r}}'$ and measured at observation point $\vec{\mathbf{r}}$ can be expanded in terms of spherical harmonics as

$$\frac{Q}{4\pi\epsilon_0}\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = \frac{Q}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)$$

(a) This expansion can be used to develop an expression for the electric potential created by a charge distribution $\rho(\vec{\mathbf{r}}')$ as a power series in r, the radial distance to the origin. Give a physical interpretation of the first three ($\ell = 0, 1, 2$) terms in that series when $r > r'$. [1 point]

(b) Assume that you have a system of four charges, two charges of size q located at $x = a$, $y = z = 0$, and two of size $-q$, located at $x = 2a$, $y = z = 0$. We may write this charge distribution using Dirac δ-functions:

$$\rho(\vec{\mathbf{r}}) = q\left\{\delta(x-a)+\delta(x+a)-\delta(x-2a)-\delta(x+2a)\right\}$$

Sketch the equipotentials and electric field lines for this distribution. Indicate the direction of increasing potential and the direction of the field lines in your diagram. [2 points]

(c) Using the expansion givien above, calculate the $\ell = 0$, 1 and 2 terms of the power series for the electric potential $\phi(\vec{\mathbf{r}})$ when $a < r < 2a$. [6 points]

(d) For the charge distribution given above, state how the magnitude of the **electric field** falls off as a function of r for $r >> a$, to leading order, and justify your answer. [1 point]

Spherical Harmonics

$Y_0^0(\theta,\phi) = \frac{1}{2\sqrt{\pi}}$ $\qquad$ $Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$ $\qquad$ $Y_1^{\pm1}(\theta,\phi) = \mp\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta e^{\pm i\phi}$

$Y_2^0(\theta,\phi) = \frac{1}{3}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$ $\qquad$ $Y_2^{\pm1}(\theta,\phi) = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{\pm i\phi}$ $\qquad$ $Y_2^{\pm2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi}$

Normalization:

$$\int_0^{2\pi} d\phi \int_0^{\pi} Y_\ell^m(\theta,\phi)\,Y_{\ell'}^{m'}(\theta,\phi)^*\sin\theta\,d\theta = \delta_{m,m'}\delta_{\ell,\ell'}$$