E & M Qualifier

January 2024

There are 5 problems. 4 of the 5 questions count as your grade on the exam. You may choose to answer only 4 questions. If you do answer all 5, the question with the lowest grade will be dropped and the remaining 4 questions will be used to grade the exam.

To ensure that your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

A spherical capacitor is composed of two concentric conducting spherical shells. Consider a spherical capacitor with an inner shell of radius a and an outer shell of radius b. We place a charge of $+Q$ on the outer shell and $-Q$ on the inner shell. Assume the space between the shells is vacuum.

(a) Determine the electric field everywhere (for all values of r). [2 points]

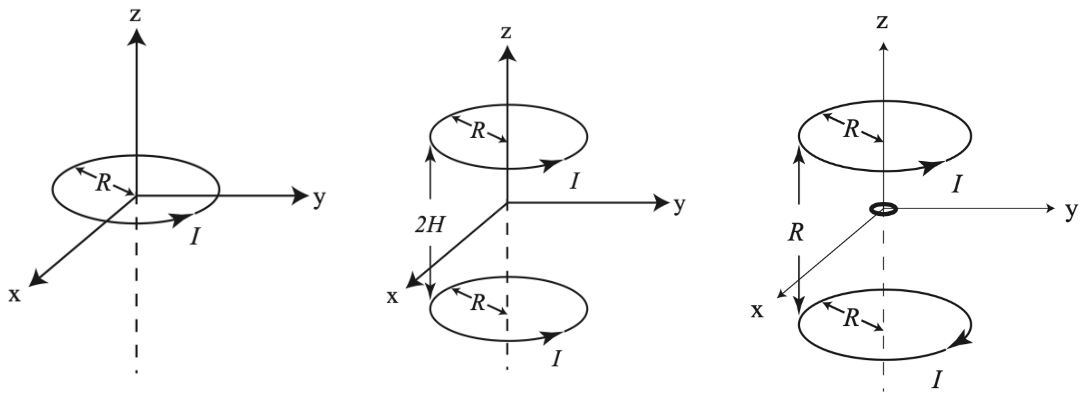
(b) Use the electric field you just calculated to find the potential difference between the spherical shells. [2 points]

(c) What is the capacitance of this capacitor? [1 point]

(d) Use the electric field you calculated to determine the energy stored on this capacitor. [2 points]

(e) Show explicitly that your result for the energy is consistent with what you'd expect of a capacitor with the capacitance, charge, and potential difference you calculated earlier. [1 point]

(f) Discuss how your answers to the previous questions would change if we inserted a dielectric with permittivity $\epsilon > \epsilon_0$ in the space between the spheres while having the same charge on the spheres (e.g., would your answers get larger, smaller, or stay the same?). Explain why those changes would happen without doing any complex calculations. [2 points]



Problem 2: Magnetostatics

(a) Consider the figure on the left panel. The loop carries a constant current I_0 and has a radius R. Calculate the magnetic field along the z-axis. [1.5 points]

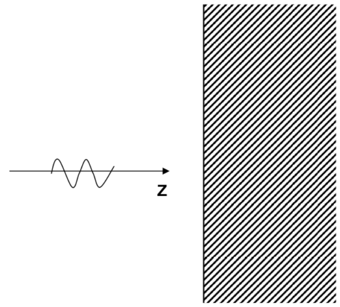
(b) Now consider the figure on the middle panel. The two loops are identical and are separated by a distance $2H$. The currents flow in the same direction. Calculate the magnetic field on the z-axis between the loops, where $z = 0$ is the midpoint between the centers of the two loops. [1.5 points]

(c) This question relates to the middle panel. Near the origin $z = 0$, the magnetic field is close to being uniform. Sketch the magnetic field lines along the $z-$axis and off-axis. [1 point]

(d) This question relates to the middle panel. Let's reverse the direction of the current in the bottom loop, while keeping the current in the top loop unchanged. Sketch the magnetic field lines along the $z-$axis and off-axis. [1 point]

(e) This question relates to the middle panel. Suppose you are given that the magnetic field along the $z-$axis in part (d) can be written as $B_z = \eta I_0 z$, where η is a constant. Determine $\partial B_x/\partial x$ and $\partial B_y/\partial y$ near the origin in terms of η and I_0. [2 points]

(f) This question relates to the right panel. Suppose you place a small loop of wire with resistance Z_0 and radius $r \ll R$ along the symmetry axis. The current in the top (bottom) loop is varied with time as $I = I_0 \cos(\omega_0 t)$ ($I = -I_0 \cos(\omega_0 t)$). Determine the magnitude and direction of the current induced in the small loop as a function of z. You can assume that the magnetic field induced by the big loops near the small loop has the form $B_z(t) = Iz = \eta I_0 z \cos(\omega_0 t)$. [3 points]



Problem 3: Electrodynamics in Metals

Consider a plane electromagnetic wave with angular frequency ω propagating the $z-$direction. It encounters a large flat piece of metal whose surface is on the $x-y$ plane. The metal has conductivity λ and magnetic permeability μ. Assume that there is no volume charge in this problem and that the speed of light is c.

(a) Write down Maxwell's Equations and use the given information to obtain an equation for the electric field $\vec{E}$ in the metal. Your equation should only involve time and space derivatives of $\vec{E}$, and possibly the parameters $\{\lambda, \mu, c\}$. [2 points]

(b) Use the plane wave solution $\vec{E} = \vec{E}_0(x, y, z)e^{i\omega t}$ and take $\vec{E}_0$ to be in the $x-$direction so that its only non-zero component is E_{0x}. Then, obtain an equation for E_{0x}. Your equation should only involve $z-$derivatives of E_{0x} and possibly the parameters $\{\lambda, \mu, c\}$. Hint: your answer should look like $\nabla^2 E_{0x} = 2ip^2 E_{0x}$, where p is a function of $\{\lambda, \mu, c\}$.. [2 points]

(c) Solve the equation for E_{0x} and obtain its $z-$dependence. Hint: Use an ansatz of the form $E_{0x} = Ae^{kz} + Be^{-kz}$, where $k^2 = 2ip^2$. Comment on the physically acceptable values of the constants A and B. [2 points]

(d) Write down the full solution of $\vec{E}$, including its $z-$ and $t-$dependence. Show that the amplitude of the electric field decays in the metal, with the characteristic decay length (penetration depth) p. [2 points]

(e) Consider a radio wave with frequency $f = 10^5$ Hz incident on copper with $\lambda = 5.9 \times 10^7 \Omega^{-1} m^{-1}$, $\mu = 1$, $\mu_0 = 4\pi \times 10^{-7}$. Determine the penetration depth p. [2 points]

Possibly useful formulae: For any vector $\vec{a}$,

$$\nabla \times \nabla \times \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a} \ . \tag{1}$$

Problem 4: Electromagnetic Radiation

A plane electromagnetic wave propagates along the $+z$ direction, and is linearly polarized in the x direction. The amplitude of the electric field is E_0 and the angular frequency of the wave is ω. Assume that the propagation occurs through vacuum and that the speed of light is c.

(a) Write expressions for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ as functions of spatial coordinates and time. Your answer must be expressed in terms of the set of parameters $\{E_0, \omega, c\}$ [2 points]

(b) For the rest of the problem, you may assume time-averaged quantities. Recall that $\langle \sin^2(\omega t) \rangle = 1/2$, where $\langle\rangle$ denotes the time average.

Determine the power per unit area carried by the wave through a surface normal to the direction of propagation. The Poynting vector is given by [3 points]

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})\,. \tag{2}$$

(c) The wave is incident on an electron (mass m, charge e) whose average position is at the origin. Calculate the total average scattered power radiated away by the electron. The Larmor formula gives the power P radiated by a point charge q that has an acceleration a: [3 points]

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3}\,. \tag{3}$$

(d) The total scattering cross section σ is the ratio of the scattered power from the electron to the incident power per unit area. Compute σ and express it in terms of the classical radius of the electron given by [2 points]

$$r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mc^2}\,. \tag{4}$$

Problem 5: Green Functions

The Laplace equation for the electric (scalar) potential is given by $\vec{\nabla}^2\Phi = 0$. Let us denote $F(\vec{x}, \vec{x}')$ as its solution where $\vec{x}$ points to the observation point and $\vec{x}'$ points to source points. Suppose instead we want the solution to the Poisson equation which includes the source charge density $\rho(\vec{x})$: $\vec{\nabla}^2\Phi = -\rho/\epsilon_0$.

(a) Replace the source term by a point distribution $\vec{\nabla}^2 G(\vec{x}, \vec{x}') = -4\pi\delta^3(\vec{x} - \vec{x}')$. What is the solution to this equation? (*Hint: It contains two terms* [2 points].

(b) Show that $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x}, \vec{x}')d^3x'$ is a solution to the Poisson equation. [2 points]

(c) For boundary value problems, we can also include boundary conditions by applying to Green's theorem: $\Phi = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x}, \vec{x}')d^3x' + \frac{1}{4\pi}\int_S \left[G(\vec{x}, \vec{x}')\frac{\partial\Phi}{\partial n} - \Phi(\vec{x}')\frac{\partial G(\vec{x}, \vec{x}')}{\partial n'}\right] da'$. What is the condition on G for Dirichlet boundary conditions where Φ is specified on the bounding surface S? Write the solution as an integral equation in this case. [3 points]

(d) What is the condition on G when Neumann boundary conditions $\frac{\partial\Phi}{\partial n}$ are specified on S? Write the solution for Φ in this case. (Assume Φ averaged over S is zero.) [3 points]