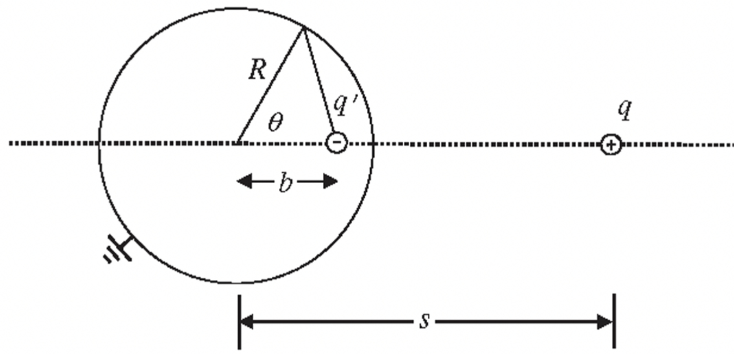
E & M Qualifier

August 2023

There are 5 problems. 4 of the 5 questions count as your grade on the exam. You may choose to answer only 4 questions. If you do answer all 5, the question with the lowest grade will be dropped and the remaining 4 questions will be used to grade the exam.

To ensure that your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
    (a) the first number is the problem number,
    (b) the second number is the page number for **that** problem (start each problem with page number 1),
    (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.



Problem 1: Electrostatics

A point charge q is placed outside a thin grounded conducting sphere of radius R centered at the origin. The real source charge is located on the z-axis at $\vec{s} = s\hat{z}$. This system can be simplified with the method of images.

(a) Find the electric potential $\Phi(\vec{r})$ at $\vec{r}$ of this system in terms of the source charge q at $\vec{s} = s\hat{z}$ and an image charge q' at $\vec{b} = b\hat{z}$. [2 points]

(b) Apply boundary conditions for $\theta = 0$ and $\theta = \pi/2$, find the location of the image charge. [4 points]

(c) What is the direction and magnitude of the electric force $\vec{F}$ on q in terms of q, R and s? [3 points]

(d) Determine the asymptotic behavior of the force $|\vec{F}|$ on q. [1 points]

Problem 2: Magnetostatics

Consider a spherical shell of radius R that carries a uniform charge density σ. It rotates about the z-axis with angular frequency ω. We will be interested in calculating the magnetic field at the center of the spherical shell.

(a) Let's first try to find the magnitude of the magnetic field using dimensional analysis. Clearly state the dimensions of the following quantity: magnetic field over the vacuum permeability B/μ_0. State your reasons or show your derivation. Then, state the dimensions of each parameter in the set $\{R, \omega, \sigma\}$ and determine a combination of these parameters that match the dimensions of B/μ_0. Finally, use your results to write down the magnitude of the magnetic field in terms of the set $\{\mu_0,\ R,\ \omega,\ \sigma\}$, up to an undetermined dimensionless constant (let's call it α). [2 points]

Dimensional analysis is powerful enough to give the parametric dependence of the answer, but not numerical pre-factors. The rest of the problem is aimed at finding the undetermined dimensionless constant α.

(b) Introduce spherical coordinates and consider a ring located on the surface of the sphere at angle θ, with differential angular width $d\theta$. Derive the current of the ring. Your answer should be in terms of $\sigma, \omega, R, d\theta$, and possibly a function of θ. [2 points]

(c) Determine the magnitude and direction of the magnetic field $d\mathbf{B}$ produced by this ring element at the center of the shell. Your answer should be in terms of $\mu_0, \sigma, \omega, R, d\theta$, and possibly a function of θ. [4 points]

(d) Integrate over θ to find the total magnitude and direction of the magnetic field at the center of the shell. [1.5 point]

(e) Compare the final answer with your dimensional analysis and read off the value of α. [0.5 points]

Problem 3: Energy and Capacitors

A spherical capacitor is created with two concentric spherical conducting shells. The inner sphere has a radius R_1, and the outer sphere has a radius R_2. We distribute a total charge of $-Q$ on the inner sphere and a total charge of Q on the outer sphere. Assume the space between the spheres is empty.

(a) Find the electric field as a function of r, the distance from the center of the spheres. Give your answer for all r. [2 points]

(b) Use your result from part (a) to determine the potential difference between the spheres. [2 points]

(c) What is the capacitance of this capacitor? [1 point]

(d) Use your result from part (a) to determine the electric energy density between the spheres. [2 points]

(e) Use your result from part (d) to determine the total energy of the capacitor. [2 points]

(f) Show explicitly that your results are equivalent to the general relationship for a capacitor, $U = \frac{1}{2}CV^2$. [1 point]

Problem 4: Electromagnetic Potentials

An insulating circular ring of radius a lies in the xy plane centered at the origin. It carries a linear charge density $\lambda = \lambda_0 sin\varphi$ where λ_0 is constant and φ is the azimuthal coordinate. The ring is set spinning at a constant angular velocity ω about the z - axis.

(a) Find the electric dipole moment $\vec{p}(\vec{r}, t)$ of the ring. Assume that at $t = 0$ the dipole moment points along the positive y-axis, and that it spins counter-clockwise. [3 points]

(b) Show that the electric dipole contribution to the retarded magnetic vector potential generated by the spinning ring, at a distance $r >> a$, is given by [4 points]

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{1}{r}\frac{\partial}{\partial t}\vec{p}(t - \frac{r}{c}) \tag{1}$$

Plug in the time dependence from part (a) to simpliffy this expression.

(c) Find the retarded electric and magnetic fields at the same distance $r >> a$ [2 points]

(d) The insulating ring is replaced with a similar one made of copper wire. A current $I(t) = I_0 cos(\omega t)$ flows through the ring. (I_0 = constant). What is the leading (i. e. largest) term in the retarded magnetic vector potential at the same distance r? [1 point]

Helpful formula:

$$|\vec{r} - \vec{r'}| \simeq r - \frac{\vec{r'} \cdot \vec{r}}{r} \quad if \quad r' \ll r \tag{2}$$

Problem 5: Motion of Charges

Consider a particle of mass m and charge q moving in an electromagnetic field. The Hamiltonian describing the motion is given by

$$H = \frac{1}{2m}\left(\vec{p} + \frac{q\vec{A}}{c}\right)^2 + q\phi \tag{1}$$

where c is the speed of light, $\vec{p}$ is the particle momentum, $\vec{A}$ is the vector potential, and ϕ is the scalar potential.

(a) Suppose you are given that the vector potential can be written as

$$\vec{A} = \frac{1}{2}\left(\vec{r} \times \vec{C}\right) \tag{2}$$

where $\vec{C}$ is a constant vector and $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Determine the units of $\vec{C}$. State what $\vec{C}$ represents and justify. [3 points]

(b) In the particular case where $\phi = 0$ and $\vec{C} = -C_0\hat{z}$, show that the Hamiltonian can be separated into a term describing the linear momentum in the z-direction p_z, a term describing the angular momentum in the z-direction L_z, and a term containing the $\hat{x}$ and $\hat{y}$ components. [3 points]

(c) Describe the motion of the particle if the initial velocity is $\vec{v} = v_0\hat{z}$. [1 point]

(d) Describe the motion of the particle if the initial velocity is $\vec{v} = v_0\hat{x}$. [1 point]

(e) What is the trajectory of the particle for an arbitrary initial velocity? [2 points]

Some possibly useful identities involving arbitrary vectors $\mathbf{A}, \mathbf{B}$ and arbitrary functions f, g are given below.

$$\nabla(fg) = f(\nabla g) + g(\nabla f)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$