E & M Qualifier

August 2021

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1

A solid sphere of radius R has a polarization (in spherical coordinates) of $\mathbf{P}(\mathbf{r}) = \alpha r^2 \hat{\mathbf{r}}$

(a) Determine the bound charge density ρ_b and the bound surface charge density σ_b for this sphere [3 points]

(b) Using your results from part (a), determine the electric field inside the sphere [3 points]

(c) Using your results from part (a), determine the electric field outside the sphere [2 points]

(d) Determine the electric potential at the center of the sphere relative to a point at infinity [2 points]

Problem 2

Consider an infinite slab with a charge per unit volume ρ_0. The slab fills the region $|z| \leq \ell/2$, where a position vector is given by $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ in a standard Cartesian coordinate system. It moves with constant velocity $\vec{\mathbf{v}} = v_0\,\hat{\mathbf{i}}$.

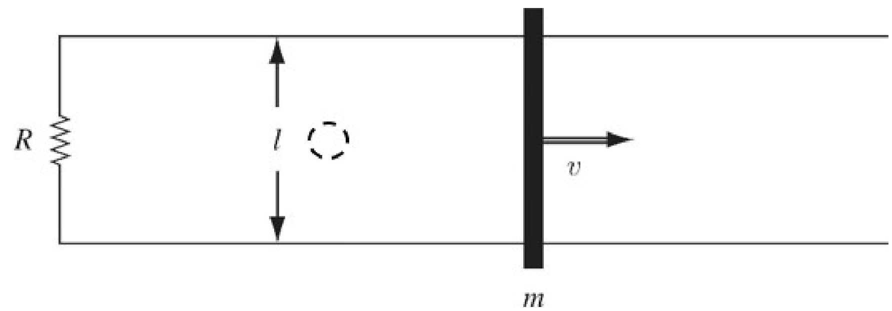
(a) Use Ampere's law or the Ampere-Maxwell equation to determine the magnetic field in all points in space outside the slab, carefully explaining your reasoning. [3 points]

(b) Use Ampere's law or the Ampere-Maxwell equation to determine the magnetic field in all points in space inside the slab, carefully explaining your reasoning. [3 points]

(c) Determine (either by integration or by careful consideration of your above answers) a magnetic vector potential in all regions of space for this problem, consistent with your answers to parts (a) and (b). [4 points]

Problem 3

A metal bar of mass m slides without friction on two parallel conducting rails that are a distance l apart. A resistor with resistance R is connected across the rails, and a uniform magnetic field $\mathbf{B}$, pointing into the page, fills the entire region. An applied force started the bar with a speed v_0 at time $t = 0$ and then was turned off. At the moment shown in the figure, the bar is moving to the right with a speed of $v(t)$. For all parts of this question, express your answer in terms of the given quantities.



(a) Find the current through the resistor. [2 points]

(b) Find the Lorentz force acting on the bar due to the current flow. In what direction is the Lorentz force? [1 point]

(c) Find the bar's speed v at a time $t(> 0)$. [2 points]

(d) Find the power lost to the resistor as a function of time. How much energy is delivered to the resistor from $t = 0$ to $t = \infty$? [2 points]

(e) Suppose a small conducting loop is placed near the center of the rectangular circuit, as shown schematically in the figure as the dashed circle. Will a current be induced in the small loop as the bar slides to the right? If so, what is the direction of the current? [1 point]

(f) Estimate the mutual inductance between the small loop and the rectangular circuit. For your estimate, assume that the small loop is infinitesimally small with an area of dA, the vertical segments of the rectangular circuit can be neglected, and the horizontal segments fo the rectangular circuit are approximately infinitely long. [2 points]

Problem 4

In a conductive medium, the plane wave solution (in complex form) for a wave propagating in the z direction is given by:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 e^{i(kz-\omega t)} \qquad \text{and} \qquad \vec{\mathbf{B}} = \vec{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

where k is the wave vector with complex magnitude $k = k_r + ik_i$.

(a) Discuss the physical meaning of the complex wave vector on the magnitudes of the electric and magnetic fields as well as on the relative phase between the electric and magnetic fields of the plane wave. Use equations to justify your answer. [2 points]

(b) For a conducting material with conductivity σ, the real and imaginary parts of the wave vector are

$$k_r = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \qquad \text{and} \qquad k_i = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

Show that the phase velocity of the electromagnetic wave in a "poor" conducting material, i.e. $\sigma \ll \epsilon\omega$, can be approximated by

$$v \simeq \frac{c}{n}\left[1+\frac{1}{4}\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{-1/2},$$

where $n = \sqrt{(\epsilon\mu)/(\epsilon_0\mu_0)}$ is the real part of the refractive index of the material. [2 points]

(c) Show that the skin depth in the metal is $(2/\sigma)\sqrt{\epsilon/\mu}$ in the limit of a poor conductor ($\sigma \ll \omega\epsilon$) and $\lambda/2\pi$ (where λ is the wavelength in the conductor) in the limit of a good conductor ($\sigma \gg \omega\epsilon$). [3 points]

(d) Calculate the electric and magnetic contributions to the time-averaged energy density of the plane wave in the conducting medium and show that the magnetic contribution always dominates. [3 points]

Problem 5

In the following, $F^{\mu\nu}$ represents the four-dimensional field tensor. $G^{\mu\nu}$ represents the dual tensor related to $F^{\mu\nu}$ by the substitution $\mathbf{E}/c \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}/c$. The proper acceleration α^μ is defined as the four-vector

$$\alpha^\mu = \frac{d\eta^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}. \tag{1}$$

The four-velocity is η^μ. The four-momentum is p^μ. The ordinary 3-dimensional velocity and acceleration are denoted by $\mathbf{u}$ and $\mathbf{a}$, respectively.

(a) Explicitly compute the fundamental tensor invariants $F^{\mu\nu}F_{\mu\nu}$ and $F^{\mu\nu}G_{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$. [2 points]

(b) Find the components of the proper acceleration, α^0 and $\boldsymbol{\alpha}$, in terms of $\mathbf{u}$ and $\mathbf{a}$. [2 points]

(c) Express $\alpha_\mu\alpha^\mu$ in terms of $\mathbf{u}$ and $\mathbf{a}$. [2 points]

(d) Show that $\eta^\mu\alpha_\mu$=0. [2 points]

(e) Write the Minkowski force, defined as

$$K^\mu = \frac{dp^\mu}{d\tau}. \tag{2}$$

in terms of the proper acceleration α^μ. Is this the same as Newton's law $\mathbf{F} = m\,\mathbf{a}$? Why or why not? What is the difference, if any? [2 points]

Problem 6

A metal cylinder of radius a and height L is placed on the $x-y$ plane so that the z-axis measures its height. The sides and bottom plate are grounded ($\Phi = 0$) while the top plate, separated from the remaining cylinder by a thin insulating layer, is set at a potential $\Phi(\rho) = V(\rho)$ (no azimuthal ϕ dependence). The electric potential inside the cylinder is expected to be of the for $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$ from separation of variables.

(a) The solutions $Q(\phi) \sim e^{\pm im\phi}$ where m is a constant of separation. What form of function do you expect $Q(\phi)$ to take and why? [1 point]

(b) The form of the functions $Z(z) \sim e^{\pm kz}$ where k is another separation constant. What form of function do you expect $Z(z)$ to take and why? [1 point]

(c) The form of the radial functions is found to be Bessel functions $R(\rho) \sim J_m(k\rho)$ and $N_m(k\rho)$. What form of radial functions $R(k\rho)$ do you expect and why [2 points]

(d) Sketch two of the radial functions as a function of ρ [2 points]

(e) Write the expected solution for $\Phi(\rho, z)$ as a power series that obeys the boundary conditions at $\rho = a$ and at $z = 0$. [2 points]

(f) How would you determine the series coefficients using the boundary condition at $Z = L$? stating in words is sufficient. [2 points]