Classical Mechanics and Statistical Mechanics/Thermodynamics Qualifier Exam

January 2026

The exam consists of five problems but only four problems will count for your grade. If you choose to solve all five problems, the problem that you score the least number of points on will be discarded. Attempt at least four problems as partial credit will be given.

To receive full credit, you must show your work.

Please adhere to the following:

- Please use only the blank answer paper provided.
- Please use only the reference material supplied.
- Please use only one side of the answer paper.
- Please put your alias (and NOT your "real" name) on every page.
- After you have completed a problem, please put three numbers on every page used for that problem:
  1. The first number is the problem number.
  2. The second number is the page number for that problem (please start each problem with page number "1").
  3. The third number is the total number of pages you used to answer that problem.
- Please do not staple your exam nor the individual problems.

Problem 1:

A particle of mass m with position vector $\vec{r}$ (components x, y, and z; $r = |\vec{r}|$) moves under the force $\vec{F}$,

$$\vec{F} = -\frac{\lambda}{r^3}\hat{r}. \tag{1}$$

Here, λ is a positive constant and $\hat{r}$ denotes the unit vector, $\hat{r} = r^{-1}\vec{r}$.

(a) (1 point) While the particle can move in all three dimensions, the motion is confined to a plane. Why?

(b) (2 points) Find the Lagrangian in polar coordinates.

(c) (3 points) Find any conserved quantities.

(d) (1 point) Find the effective potential.

(e) (2 points) Sketch the effective potential for all relevant combinations of physical parameters.

(f) (1 point) Under what conditions (if any) does the system support bound orbits?

Problem 2:

A ring (mass M and radius R) is supported from a pivot point O on the ring. The ring can rotate freely in its own vertical plane [i.e., the ring can rotate about the z-axis (axis that goes through O and that is perpendicular to the plane that the ring lies in)]. A bead (mass m) slides about the ring without friction (see the figure for a schematic; note that the mass m is constrained to be located on the ring).

(a) (2 points) Using the coordinates shown in the figure, derive the Lagrangian for this system.

(b) (3 points) Find the equations of motion.

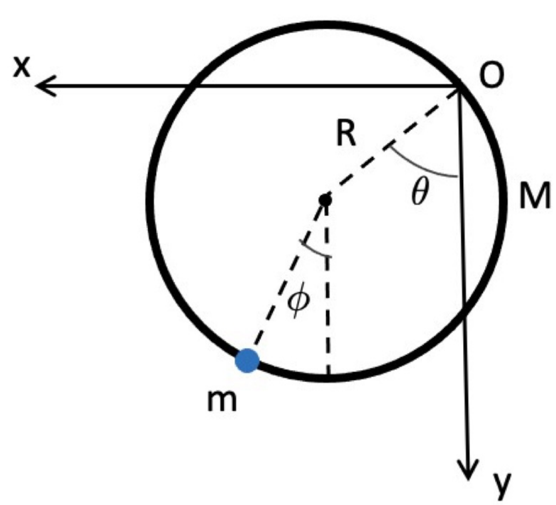
(c) (2 points) Assuming small oscillations, simplify the equations of motion.

(d) (3 points) Assuming small oscillations, set up the characteristic equation that would allow you to determine the normal mode eigen frequencies (you do NOT have to solve the characteristic equation).

Possibly useful information:

- The moment of inertia I of a ring with radius R and mass M about an axis that goes through the center of the ring and is perpendicular to the plane that the ring lies in is given by $I = MR^2$.

- $\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\phi - \theta)$.



Problem 3:

Parts (a), (b), and (c) of this problem consider a single particle with generalized coordinate q and generalized momentum p. The force is assumed to be conservative.

(a) (1 point) Assuming the units of q are known, what are the units of p?

(b) (3 points) Assume that the energy E of the particle under consideration is constant, sketch a phase space trajectory that corresponds to periodic motion and sketch a second phase space trajectory that corresponds to unbound motion. Provide a concrete physical example for both cases.

(c) (3 points) Consider a phase space trajectory for energy E_1 and a second phase space trajectory for energy E_2, where $E_1 \neq E_2$. Are the two phase space trajectories allowed to cross? If yes, under which conditions? If no, why not?

(d) (3 points) What is Liouville's theorem? Explain how it relate to parts (a)-(c) of this problem.

Note: Since several parts of this problem are conceptual, please make sure to clearly explain your reasoning. One word answers will not be given any credit.

Problem 4:

An engine is going through the cycle shown in the figure. The working medium is the ideal monoatomic gas. AD and BC are adiabatic processes, while AB and DC are isochoric processes (i.e., processes at constant volume).

(a) (1 point) What direction, clockwise or anti-clockwise, does the engine have to cycle to generate positive work?

(b) (2 points) Which segments correspond to (use the notation "start $\rightarrow$ end" to indicate the direction)

  i. heat input to the system (Q_{in})?

  ii. heat output from the system (Q_{out})?

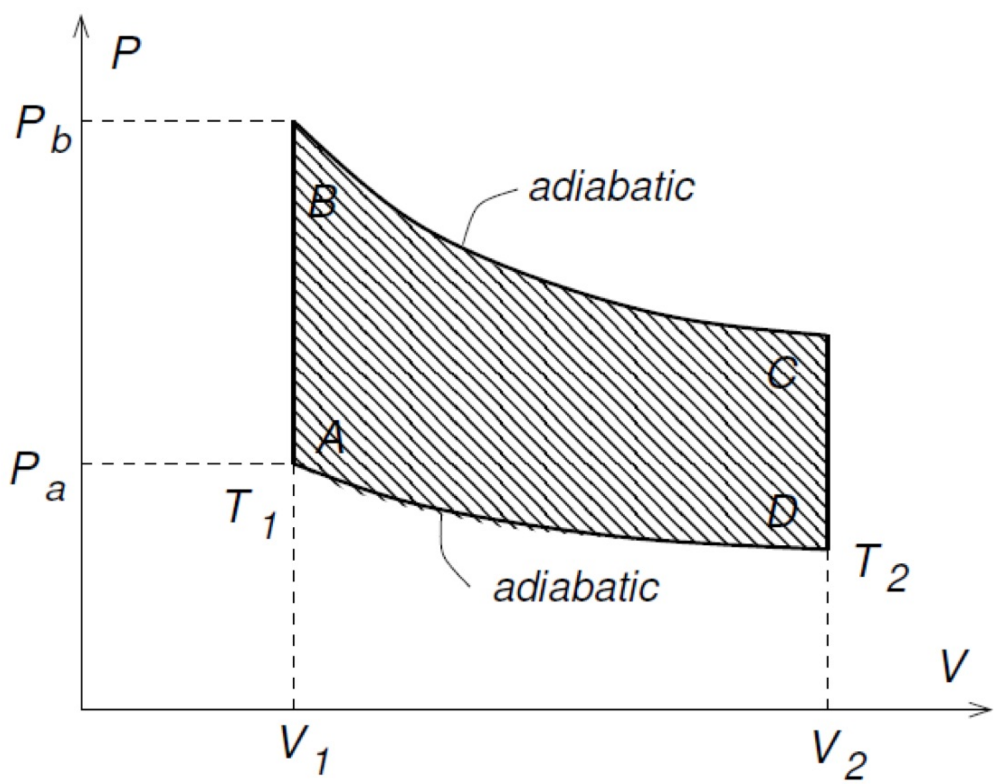
  iii. work done on the system (W_{in})?

  iv. work done by the system (W_{out})?

(c) (2 points) Find Q_{in} and Q_{out} for the process.

(d) (2 points) What is the work done in one full (i.e., closed) cycle?

(e) (3 points) What is the efficiency of this engine? Express your answer using only V_1, V_2, and numerical constants.



Problem 5:

Consider a gas of spin-zero bosons with single-particle dispersion relation (also referred to as single-particle energy) $\epsilon_p = \alpha|\vec{p}|^s$, where $\vec{p}$ is the momentum of the boson ($p = |\vec{p}|$) and α and s are positive constants with appropriate units, in a three-dimensional container of volume V; note that s does not denote the spin.

(a) (2 points) Treat the gas in the grand canonical ensemble and determine the partition function for the gas.

(b) (3 points) Find an expression for the mean number of particles as a function of the temperature T and the fugacity z, $z = e^{\beta\mu}$, where $\beta = (k_B T)^{-1}$ (k_B denotes the Boltzmann constant) and μ is the chemical potential.

(c) (2 points) Prove that the density of states $g(\epsilon)$ of the gas is proportional to

$$\epsilon^{\frac{3-s}{s}}. \tag{2}$$

(d) (3 points) Find the equation of state for the gas. Write your answer as an integral over all possible values of the single-particle energy.