Classical Mechanics and Statistical/Thermodynamics

January 2024

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.
7. **There are 5 problems but only 4 problems will count to your grade. If you choose to solve all 5, the problem on which you score the least will be discarded. Please attempt at least four problems as partial credit will be given.**

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}$$

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\; e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\; e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\;dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{or} \quad \log(n!) \approx n\log(n) - n$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Handy Taylor Series:

$$\log(1+x) = \sum_{n=1}^\infty (-1)^{n+1}\frac{x^n}{n}$$

$$\log(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant $\mathrm{K} = 8.998 \times 10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85 \times 10^{-12} \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A	electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg	Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K	Planck's constant: $\hbar = 6.63 \times 10^{-34}$m^{2}kg/s
speed of light: $c = 3.00 \times 10^8$m/s	Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm} \cdot \mathrm{mol}^{-1}\mathrm{K}^{-1}$

Question 1: *For the following problem studying projectile motion near the surface of the earth, assume an (x,y) coordinate system with gravity acting vertically in the $-y$ direction. Neglect air resistance throughout.*

A projectile is launched at a $45°$ angle with respect to the horizontal and has an initial kinetic energy E_0. When the projectile reaches its maximum height it instantaneously explodes into two fragments with masses m_1 and m_2, respectively. The explosion *adds an additional amount of mechanical energy E_0 to the system* (i.e., the two fragments). After the explosion, the velocity of the fragment with mass m_1 is directed straight down, in the $-y$ direction. For the following questions give all answers in terms of E_0, m_1 and m_2.

(a) What was the initial speed of the projectile? (1.5 points)

(b) Obtain expressions for the velocities $\vec{v}_1$ and $\vec{v}_2$ of the masses m_1 and m_2 immediately after the explosion. (4 points)

(c) Show that the maximum value of the ratio of the two masses is $m_1/m_2 = 2$. Explain your answer. (1.5 points)

(d) Assuming $m_1 = 2m_2$, how far will each of the two fragments land on the ground from where they were first launched? (3 points)

Question 2: Two identical blocks, each of mass m, sit on a frictionless horizontal surface. The blocks are connected by a spring with force constant k and equilibrium length ℓ_0. In the following you may assume their motion is confined to one dimension.

(a) Defining ℓ as the distance between the masses, show that their relative motion can be described by a one-dimensional Lagrangian,

$$\mathcal{L}(q, \dot{q}, t) = \frac{\mu}{2}\dot{q}^2 - \frac{\mu\omega^2}{2}q^2,$$

where we have defined the reduced mass $\mu = m/2$ for the composite system, $\omega = \sqrt{k/\mu}$ and $q = \ell - \ell_0$ is the generalized co-ordinate. What characteristic of this Lagrangian enables you to deduce that energy is conserved in this system? (3 points)

(b) Derive the corresponding Hamiltonian $\mathcal{H}(q, p)$ for the system where q and p are the canonical position and momentum. In general, when is the Hamiltonian equal to the energy of the system? (2 points)

(c) For a transformation of the form,

$$Q = C\left(q - \frac{p}{\mu\omega}\right),$$
$$P = C\left(q + \frac{p}{\mu\omega}\right),$$

find an expression for C such that this is a canonical transformation. (3 points)

(d) Find the generating function of the second kind, $F_2(q, P, t)$ (see table below), that is associated with the transformation in part (c). Use this to obtain the transformed Hamiltonian $\tilde{\mathcal{H}}(Q, P)$ in terms of the new canonical co-ordinates. (2 points)

Generating function	Derivatives	
$F = F_1(q, Q, t)$	$p = \frac{\partial F_1}{\partial q}$	$P = -\frac{\partial F_1}{\partial Q}$
$F = F_2(q, P, t) - QP$	$p = \frac{\partial F_2}{\partial q}$	$Q = \frac{\partial F_2}{\partial P}$
$F = F_3(p, Q, t) - qp$	$q = -\frac{\partial F_3}{\partial p}$	$P = -\frac{\partial F_3}{\partial Q}$
$F = F_4(p, P, t) + qp - QP$	$q = -\frac{\partial F_4}{\partial p}$	$P = \frac{\partial F_4}{\partial P}$

Question 3:

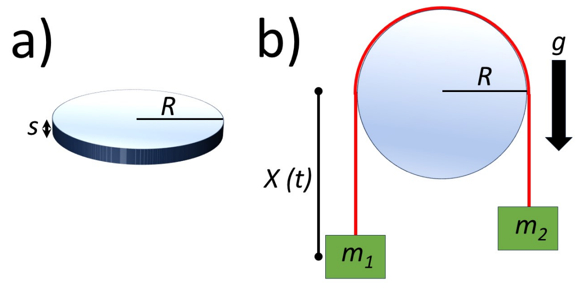

Figure 1: a) The circular disc of radius R and vanishing thickness s. b) The disc can function as a frictionless pulley, upon which a string connecting two masses is placed. Gravity acts along the vertical direction as indicated.

Consider a circular disc of radius R and infinitesimal thickness s [see Fig. 1 a)]. The density of the disc varies with the radial distance from the center as $\rho(r) = kr^2$ where $k > 0$ is a constant.

(a) Show that the moment of inertia of the disc about its axis of radial symmetry can be expressed in terms of the total mass M of the disc and the radius as,

$$I = \frac{2}{3}MR^2.$$

Note: You should obtain an expression for the total mass of the disc M as a function of the radius R and other constants as part of your solution. (1.5 points)

(b) Suppose that the disc is rotating about its axis of radial symmetry at a constant angular velocity. Using your result in (a), give an expression for the kinetic energy T of the disc in terms of M and the velocity v of a point at the outer (radial) edge of the disc. (1 point)

For the remainder of the problem, we assume that the disc is converted to function as a frictionless pulley. A string of fixed length L is run across the pulley and a pair of blocks with mass m_1 and m_2 are affixed to each end [see Fig. 1 b)]. We will assume that the string is massless and does not slip on the pulley, while the masses of the blocks satisfy $m_1 > m_2$. The blocks only move in the vertical direction and are subject to gravity.

(c) Obtain a Lagrangian describing the motion of the pulley-mass system in terms of the generalized co-ordinate x that is defined to be the vertical distance between the center of the pulley and the center of the block of mass m_1. (2.5 points)

(d) By writing down the Euler-Lagrange equation for x, obtain an expression for the vertical acceleration of the block of mass m_1. Express your result in terms of m_1, m_2, M and g. (2.5 points)

(e) By computing the tensions T_1 and T_2 of the string just above the blocks of mass m_1 and m_2, respectively, compute the torque acting on the pulley. (2.5 points)

Question 4:

(a) In your own words, compare and contrast the microcanonical and canonical ensembles. (2.5 points)

For the remainder of this question, consider a classical non-interacting gas of N distinguishable atoms with mass m confined to a length L in one dimension. In addition to the single-particle kinetic energy $p^2/(2m)$ where p is the momentum, each particle can be in one of two internal states: a ground-state with energy 0 and an excited state with energy $\epsilon > 0$.

(b) Explain why the canonical partition function of the N particle gas can be written,

$$Q_N(T,V,N) = [Q_1(T,V)]^N,$$

where $Q_1(T,V)$ is the partition function of a single atom. (1.5 points)

(c) Using the result of (b), show that

$$Q_N(T,V,N) = \frac{L^N}{h^N}(2\pi m k_B T)^{N/2}\left(1 + e^{-\frac{\epsilon}{k_B T}}\right)^N.$$

Hint: Useful integrals can be found page 2 of the exam paper. (3 points)

(d) Finally, use the partition function to show that the specific heat of the gas is

$$C_V = N k_B \left[\frac{1}{2} + \frac{\epsilon^2}{(k_B T)^2}\frac{e^{\frac{\epsilon}{k_B T}}}{(1 + e^{\frac{\epsilon}{k_B T}})^2}\right].$$

Comment on the behaviour of the specific heat at low temperatures. (3 points)

Question 5: Conduction electrons in a metal can be well described as a 3D gas of spin-1/2 fermions (i.e., fermions with spin up or down) with density $n = N/V$ where N is the total number of electrons in a volume V. The electrons have single particle energies,

$$\epsilon_{\vec{k}} = \frac{\hbar^2 |\vec{k}|^2}{2m},$$

where $\vec{k}$ is the wavenumber and m the electron mass.

(a) Show that the Fermi energy can be expressed as,

$$\epsilon_F = \frac{\hbar^2}{2m} \left(3\pi^2 n\right)^{2/3}.$$

Qualitatively describe the ground-state configuration of the gas in relation to the Fermi energy. (2 points)

(b) Show that the ground-state energy density of the electron gas is,

$$\frac{E_0}{V} = \frac{3}{5}(3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{5/3}.$$

(2 points)

Real electrons interact via Coulomb repulsion and this can lead to an imbalance in the number of spin up and down electrons in the ground-state. In the context of our electron gas, this interaction can be captured by an effective spin-spin coupling that favours states with spins arranged in parallel, corresponding to a potential energy term,

$$U = \alpha \frac{N_+ N_-}{V},$$

added to the Hamiltonian. Here, N_+ and $N_- = N - N_+$ are the numbers of electrons with spin up and down, respectively, and α is some constant that characterizes the strength of the interaction.

(c) Show that when the the density of spin up and down electrons is very close, i.e., $n_+ = N_+/V = n/2 + \delta$ and $n_- = N_-/V = n/2 - \delta$ with $\delta \ll n$, the total kinetic energy density can be approximated to second order in δ as,

$$\frac{E_{\text{kin}}}{V} = \frac{E_0}{V} + \frac{4}{3}(3\pi^2)^{2/3} \frac{\hbar^2}{2m} \frac{\delta^2}{n^{1/3}},$$

where E_0/V is the energy density of the non-interacting ground-state of part (a). (3 points)

(d) By also obtaining an expression for the interaction energy in terms of δ and n, show that the electron gas can lower its total energy by spontaneously magnetizing for a sufficiently strong interaction $\alpha > \alpha_c$ where the critical interaction strength is,

$$\alpha_c = \frac{4}{3}(3\pi^2)^{2/3} \frac{\hbar^2}{2m} \frac{1}{n^{1/3}}.$$

Discuss how interactions have changed your result for the ground-state in relation to the non-interacting case of parts (a) and (b). (3 points)