Classical Mechanics and Statistical Mechanics/Thermodynamics
January 2021

Please adhere to the following:

- Please use only the blank answer paper provided.
- Please use only the reference material supplied.
- Please use only one side of the answer paper.
- Please put your alias (and NOT your real name) on every page.
- After you have completed a problem, put three numbers on every page used for that problem:
    1. The first number is the problem number.
    2. The second number is the page number for that problem (please start each problem with page number "1").
    3. The third number is the total number of pages you used to answer that problem.
- Please do not staple your exam nor the individual problems.

Problem 1:

Consider a wheel with radius R and mass m. The wheel is affixed at the center of an axle with length $2r$ and can freely rotate around the axle with an angular velocity ω that points in the same direction as the axle. The axle is initially horizontal (oriented along the positive x-axis) and is hanging from a vertical massless rope that is stretched along the y-axis and connected to the axle at one of its ends (that end is located at the origin; see figure below). The system is placed in a homogeneous gravitational field. Consider all the mass of the wheel to be at the outer rim of the wheel.

(a) (1 point) If the angular velocity is such that the axle remains in the xz-plane, explain in words what will happen and why. Your explanation should include ideas such as force, torque, angular momentum, rotation, etc.

(b) (1 point) Calculate the tension in the string.

(c) (3 points) Calculate the algebraic solution for any motion of the wheel that you described in part (a).

(d) (1 point) If the wheel was rotating in the opposite direction, would your solution change? If so, explain how and why.

(e) (1 point) Assume that a constant force is applied in the $+y$-direction at the outer end of the axle (i.e., the end of the axle that is not attached to the rope). The magnitude of the force is adjusted so that it is equal to one half of the tension in the rope and such that the axle continues to lie in xz-plane. Describe in words what will happen.

(f) (3 points) Calculate the tension in the rope and the magnitude of the applied force for the situation considered in part (e). Algebraically calculate the solution for the motion described in part (e).

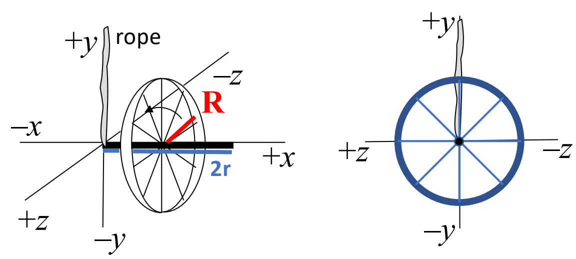

Figure 1: Schematic for Problem 1: Three-dimensional sketch and front view.

Problem 2:

A block slides frictionlessly on a horizontal surface. A circular loop is attached rigidly to the top of the block. The "block plus loop" structure has mass M and moves along the x-direction. Another mass m slides on the inside of the loop of radius R. The entire system is placed in a homogeneous gravitational field (see schematic below).

(a) (1 point) Find the coordinates of the mass m.

(b) (3 points) Find the Lagrangian of the entire system.

(c) (2 points) Derive the Euler-Lagrange equations.

(d) (1 point) Re-write the Euler-Lagrange equations assuming small oscillations of the mass m about the bottom of the loop.

(e) (3 points) Find the angular frequency of the small oscillations.

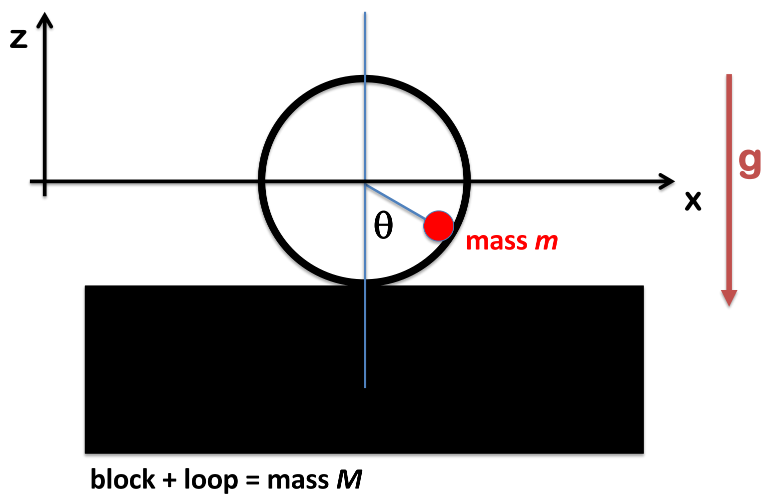

Figure 2: Schematic for Problem 2.

Problem 3:

Consider a system consisting of N doubly-charged ions and work in the regime where each doubly-charged ion can be treated as a classical point particle of mass m. The ions are confined to move along the z-axis. They interact electromagnetically (through the Coulomb force) with each other. The ions are also subjected to an external potential with potential energy in the form $V(z_j) = \epsilon |z_j|/a$, where ϵ is a positive constant with units of energy, a is a positive constant with units of length, and z_j is the position of the jth ion along the z-direction, measured with respect to the origin. The Coulomb interaction and the external potentials $V(z_j)$ $(j = 1, \cdots, N)$ are the only potential energies in the problem. Throughout the problem, assume that finite temperature effects can be neglected.

(a) (1.5 points) Write down the Lagrangian L of the system.

(b) (1 point) Denote the coordinates for which the total mechanical energy is minimal by $\vec{z}_0 = (z_{1,0}, \cdots, z_{N,0})^T$. What is the condition that determines $\vec{z}_0$? You **do not** have to determine an explicit expression for $\vec{z}_0$.

(c) (2 points) Specialize to the $N = 2$ case and determine an explicit expression for $\vec{z}_0$.

(d) (3 points) Continue with the $N = 2$ case. Taylor expand the Lagrangian L from part (a) around $\vec{z}_0$ up to second order, i.e., find an approximate "quadratic" expression for the Lagrangian.

(e) (2.5 points) You do not have to address this part mathematically; concise qualitative arguments are sufficient. As N changes from $N = 4$ to $N = 6$, the distance between the two centermost particles decreases. Explain why.

Problem 4:

A problem on gas expansion.

The initial state of monoatomic ideal gas is described by T_A, P_A, and V_A (the temperature, pressure, and volume, respectively). The gas is taken over the path $A \to B \to C$ quasi-statically, as shown in the sketch below. The volumes are related as $V_B = 1.5V_A$ and $V_C = 2V_A$.

(a) (3 points) How much work does the gas do on the path $A \to B$ and what is the change in its internal energy?

(b) (2 points) How much heat is absorbed in going from $A \to B$?

(c) (3 points) Derive the expression for the entropy change for a reversible adiabatic process $B \to C$.

(d) (2 points) If $B \to C$ is a reversible adiabatic process, find the final gas pressure and the entropy change (from the general expression obtained in part (c)).

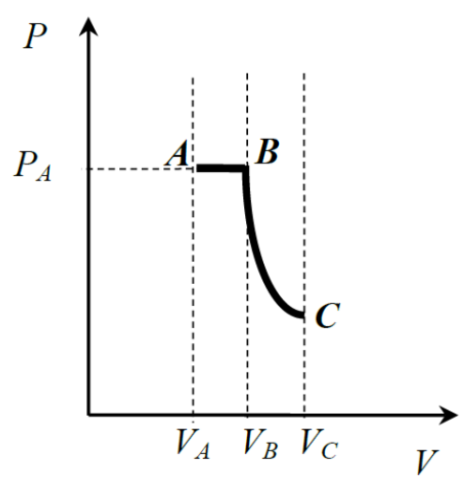

Figure 3: Schematic for Problem 4.

Problem 5:

Consider N atoms that are pinned at fixed positions (they might be arranged in a regular pattern). The atoms do not interact with each other. Each atom has four energy levels; the energies are 0, ϵ, 99ϵ, and 100ϵ, where ϵ is positive and has units of energy. The temperature T of the system is fixed.

(a) (2 points) Determine the internal energy U of the system.

(b) (2.5 points) Sketch how the internal energy U changes as a function of temperature T. Note, even though you do not have access to a "graphing device" (computer or fancy calculator), you can address this question by considering appropriately chosen limits such as the internal energy per particle approaching zero and the internal energy per particle approaching 100ϵ.

(c) (2 points) The expression you found in (a) is, given that you do not have access to a computer or fancy graphing device, not overly handy. Assuming that $k_B T \lesssim \epsilon$, find an approximate but explicit expression for the temperature. Here, k_B denotes the Boltzmann constant.

(d) (1.5 points) Consider your result from part (c). What happens to the temperature when $U \approx N\epsilon$ (but $U < N\epsilon$)? Is the expression you found in part (c) physical for $U \gg N\epsilon$? If yes, explain why. If no, explain why not.

(e) (2 points) Explain which ensemble you used to tackle this problem and why. What are the key characteristics (i.e., macrovariables) of this ensemble? Had you chosen a different ensemble, how would your results differ?

Problem 6:

Consider an ensemble of N non-interacting bosons in a two dimensional (2D) box with sides of length L.

(a) (1.5 points) Assuming that a Bose-Einstein condensate (BEC) is formed in the ground-state (e.g., energy $E = 0$) as $T \to 0$, show that the chemical potential μ obeys $|\mu|/(k_B T) \ll 1$ for $N \gg 1$. Here, k_B denotes the Boltzmann constant and T the temperature.

(b) (1.5 points) Based on the result $|\mu|/(k_B T) \ll 1$ from (a), show that, in fact,

$$\mu \approx -\frac{k_B T}{N} \tag{1}$$

for a BEC as $T \to 0$.

(c) (3.5 points) Assuming that the bosons are massless (hint: this means they should be treated as a relativistic gas), show that the critical temperature T_c for a BEC to form is:

$$T_c \approx \frac{2\hbar c}{k_B}\sqrt{\frac{3N}{L^2\pi}}. \tag{2}$$

Comment on how the critical temperature is defined before deriving Eq. (2). During your derivation of T_c, discuss how the relation in (a) has entered in any assumptions or approximations you have made.
Hint: You might find the following identities very useful,

$$\int_0^\infty \frac{x}{e^{\beta(x-\alpha)}-1}dx = \frac{1}{\beta^2}\sum_{l=1}^{\infty}\frac{e^{l\alpha\beta}}{l^2} \text{ and } \sum_{l=1}^{\infty}\frac{1}{l^2} = \pi^2/6; \tag{3}$$

you **do not** have to derive these identities.

(d) (2 points) Taking $T = T_c$ and assuming there are $N \gg 1$ of these massless bosons, use the results of (b) and (c) [e.g., Eqs. (1) and (2)] to show that $|\mu| \ll \Delta E$, where ΔE is the energy gap between the ground-state and lowest-lying excited state. Why is this result important in the context of a condensate forming?

(e) (1.5 points) If you were to perform the calculation of part (c) again but assuming the bosons are massive, would you still expect a BEC to form in 2D?
Hint: You do not actually need to redo (c), rather you should be able to highlight qualitative differences in the assumptions, e.g., density of states, that would go into your calculation.