Classical Mechanics and Statistical/Thermodynamics

January 2020

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\,\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A
electronic mass $m_e = 9.11 \times 10^{-31}$kg
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K
speed of light: $c = 3.00 \times 10^8$m/s
$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
electronic charge $e = 1.60 \times 10^{-19}$C
Density of pure water: 1.00gm/cm^3.
Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$
Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

Classical Mechanics

1. Shown below in Figure 1 are two separate triangular ramps, each with mass M, at rest on a frictionless surface. The upper ramp surface makes an angle θ with respect to the horizontal. A small block with mass m is released from rest on one of the ramps at a height h as shown in the figure. All shaded edges indicate frictionless surfaces. The block slides without friction on each of the ramps and on the surface between them; the ramps slide on the horizontal surface without friction.

Figure 1: The block and ramps all slide without friction.

(a) What is the speed of the block when it reaches the horizontal surface? (2 points)

(b) The block will reach the second ramp and then slide up it without friction. Assuming an ideal situation (no bumps or jumps as it meets the lower edge of the ramp) how high will it travel up the second ramp? (3 points)

In figure (2) the block is replaced with a solid spherical object with radius R and mass m. Only the bottom of the ramps are frictionless so that the ramps can still slide horizontally without friction, but the ball will roll without slipping down the first ramp, roll across the horizontal surface, and roll up the second ramp.

Figure 2: The ramps slide without friction; the ball rolls without slipping.

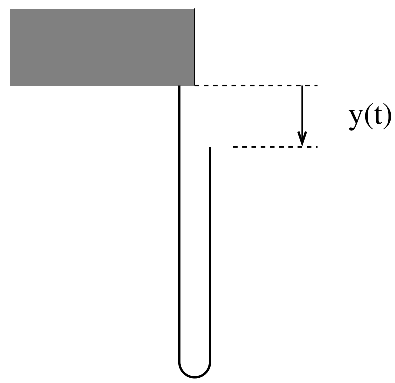
(c) When the ball travels across the horizontal surface will its speed be larger than, smaller than, or equal to the speed of the block you found in (a) above? What about its momentum? **You do not have to calculate its speed,** but you must explain your reasoning. (1 point)

(d) When the ball rolls up the second ramp, will it reach a height greater than, less than, or equal to the answer you found in (b) above? You do not have to calculate its maximum height, but you must explain your reasoning. (1 point)

(e) Return to the situation of Figure 1, in part (a) above. Write down (but do not solve) Newton's second law for the block and the ramp separately. (2 points)

(f) Show that the equations from part (e) above are not sufficient for determining the initial vertical and horizontal acceleration of the block. Provide the equation of constraint that ensures that the block slides maintains contact with ramp. (You do not have to solve these equations.) (1 point)

2. A heavy rope of mass m, and length L has uniform density and is attached at one end to the edge of a (fixed) balcony. It is initially at rest with its free end at a height equal to the point of support so that $y(t = 0) = 0$. The end of the rope is released and it falls downward due to the force of gravity. You should treat the rope as ideal, so that the portion at the left hand side is at rest, and all motion is elastic.



   (a) Determine the position of the center of mass of the rope as a function of y. (2 points)

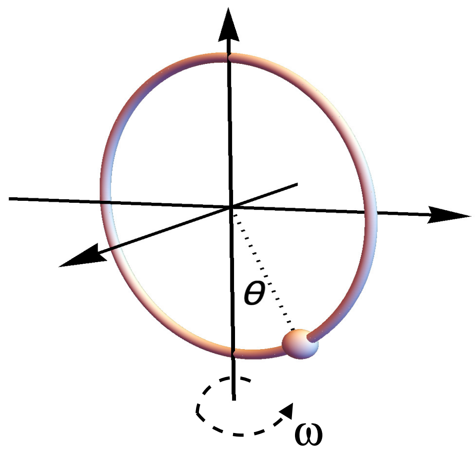
   (b) Using the above expression for the center of mass, write down the Lagrangian for the rope. (2 points)

   (c) Determine the speed at which the end of the rope falls as a function of y. (2 points)

   (d) Determine the acceleration of the end of the rope as a function of height. (2 points)

   (e) How does your answer compare to g? Explain your result. (2 points)

3. A point mass is constrained to move on a massless hoop of fixed radius a. The hoop rotates with constant angular speed ω about its vertical symmetry axis, which lies on the z-axis, but has no other motion (e.g. translational motion). The mass slides on the hoop without friction.



(a) Write down the Lagrangian for the system (1 point)

(b) Obtain the equations of motion of the point mass from the Lagrangian, assuming that the only external forces arise from gravity. (2 points)

(c) Show that if ω is greater than some critical value ω_c there can be a solution in which the particle remains stationary on the hoop at a point other than the top or bottom, but for $\omega < \omega_c$ the only stationary point for the particle is at the top or bottom of the hoop. What is the value of ω_c? (4 points)

(d) Assume $\omega < \omega_c$, and that the particle is displaced slightly from the bottom of the hoop by an angle $\theta << \pi/2$. Calculate the particle's subsequent motion in this approximation. (3 points)

Statistical Mechanics

4. A spherical bubble of n moles of ideal monatomic gas, with an initial radius $r = R_0$ forms at an initial depth $y = d$ in the ocean and ascends to the surface ($y = 0$). Consider an idealized ocean with a uniform, constant density ρ and temperature T. The bubble remains in thermal equilibrium with the surrounding water during its ascent. Assume that the density of the gas inside the bubble is insignificant compared to that of the sea water and that it does not exchange gases with the sea water as it rises. Also assume that $r << d$. The gas constant is R and the acceleration of gravity is g. Formulate your answers in terms of the variables given.

   (a) Find the equation for the radius of the bubble as a function of y. (2 points)

   (b) Find the thermodynamic work done on the bubble as it rises to the surface as a function of y. (The thermodynamic work excludes work done related to motion of the center of mass of the bubble). (3 points)

   (c) Find the equation for the change in the entropy of the bubble. (2 points)

   (d) Does the buoyant force increase, decrease, or remain the same as the bubble rises? (1 point)

   (e) Suppose the bubbles ascent was not isothermal, but perfectly adiabatic. Answer qualitatively how this would change the radius of the bubble (part a), the work done (part b), and the change in entropy (part c). Explain your reasoning. (2 points)

5. A system consists of N identical particles that can each either have an energy 0 or ϵ.

   (a) Calculate the partition function in the micro-canonical ensemble when the total energy is $E = p\,\epsilon$, with p equal to an integer less than N. (1 point)

   (b) Calculate the temperature of the system described by the partition function in (a). (2 points)

   (c) Calculate the partition function in the canonical ensemble when the temperature of the system is fixed at T. (1 point)

   (d) Calculate the average energy of the system described by the partition function in (c). (2 points)

   (e) Show how the energy and temperature in an ensemble of fixed E is related to the energy and temperature at fixed T for the system described above. (2 points)

   (f) Given a system such as this, what observables might distinguish the two ensembles described above? (2 points)

6. *Ultra-relativistic fermions*:

Consider a non-interacting ideal gas of fermions with spin 1/2 in three dimensions. The fermions have single particle energy states $E_j > 0$, labelled by some index j.

(a) Show that the grand canonical potential $\Phi(T, V, \mu)$ can be written as

$$\Phi(T, V, \mu) = -kT \int_0^\infty d\epsilon \frac{\Omega(\epsilon)}{1 + e^{\beta(\epsilon - \mu)}}, \tag{1}$$

where μ is the chemical potential, and $\Omega(\epsilon)$ is the number of states with energy smaller than ϵ, so that the density of states $\rho(\epsilon)$ is given by $\rho(\epsilon) = d\Omega/d\epsilon$. (3 points)

(b) From now on, consider highly relativistic free non-interacting fermions, so the energy of a particle with momentum $\vec{p}$ is $\epsilon_p \approx pc$ where c is the speed of light. Calculate the number of states in three dimensions

$$\Omega(\epsilon) = \frac{2V}{h^3} \int_{\epsilon_p < \epsilon} d^3p, \tag{2}$$

with fermion energy ϵ_p being smaller than ϵ. (2 points)

(c) Show that the density of states is given by $\rho(\epsilon) = 3\Omega(\epsilon)/\epsilon$. (1 point)

(d) Show that $\Phi = -E/3$, where E is the total average energy of the system. (2 points)

(e) Obtain an expression for the pressure $P(T, V, \mu)$. Your answer may involve the functions $f(z)$ and/or $g(z)$ given on the formula sheet. (2 points)