Classical Mechanics and Statistical/Thermodynamics

August 2023

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.
7. **There are 5 problems but only 4 problems will count to your grade. If you choose to solve all 5, the problem on which you score the least will be discarded. Please attempt at least four problems as partial credit will be given.**

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}$$

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{or} \quad \log(n!) \approx n \log(n) - n$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Handy Taylor Series:

$$\log(1+x) = \sum_{n=1}^\infty (-1)^{n+1} \frac{x^n}{n}$$

$$\log(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\, \frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant $K = 8.998 \times 10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85 \times 10^{-12} \mathrm{C^2/N \cdot m^2}$
$\mu_0 = 4\pi \times 10^{-7}$T m/A	electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg	Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K	Planck's constant: $\hbar = 6.63 \times 10^{-34} \mathrm{m^2 kg/s}$
speed of light: $c = 3.00 \times 10^8$m/s	Ideal Gas Constant: $R = 0.0820\, \ell \mathrm{atm \cdot mol^{-1} K^{-1}}$

Question 1: A uniform sphere with mass m and radius R is spinning with an angular velocity of ω_0 about a horizontal axis. The sphere is initially spinning just above a horizontal table but is then lowered to be brought into contact with the surface of the table. The sphere experiences a gravitational force downward and a frictional force from the table with a coefficient of kinetic friction given by μ. The moment of inertia of a uniform sphere is $I = (2/5)mR^2$.

(a) Assuming that $t = 0$ when the sphere first contacts the table, show that the sphere begins to roll without slipping at time,

$$t = \frac{2\omega_0 R}{7\mu g}.$$

(3 points)

(b) What is the translational velocity and the rotational velocity of the sphere when it first begins to roll without slipping? (2 points)

(c) How far will the sphere have travelled from where it was first placed on the table to when it begins to roll without slipping? (1 point)

(d) In general, the moment of inertia of many uniform objects rotating about their center of mass can be written as CmR^2 where C is some constant, m is the mass and R the radius of the object. This is true for, e.g., disks, spheres, etc. Show that if the sphere of this problem is replaced by a generic object with momentum of inertia $I = CmR^2$ (but follows the same sequence), the fraction of energy *remaining* when the object begins to roll without slipping is $C/(1 + C)$ and thus for the sphere, where $C = 2/5$, the fraction of energy remaining is $2/7$. (4 points)

Question 2: Consider a particle of mass m moving in one dimension, described by the Lagrangian:

$$L(x, \dot{x}) = -mv_0\sqrt{v_0^2 - \dot{x}^2} + F_0 x \ .$$

Here, $\dot{x} = dx/dt$, v_0 is a constant with units distance/time and F_0 has units of energy/distance.

(a) Determine whether each of the following is a constant of motion for the particle:

i) mechanical energy

ii) canonical momentum

Explain your answer in each case. (2 points)

(b) Show that the corresponding Hamiltonian for the system is,

$$H(x, p) = v_0\sqrt{p^2 + m^2 v_0^2} - F_0 x \ .$$

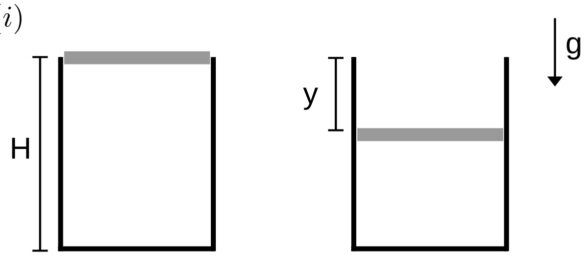
(2 points)

(c) Use $H(x, p)$ and the associated Hamilton's equations to sketch a phase portrait for the motion of the particle (in the $x - p$ plane). (3 points)

(d) Solve for the trajectory of the particle $(x(t), p(t))$ assuming the initial condition $x(0) = p(0) = 0$. Discuss the behaviour of the particle at long times. (3 points)

Question 3: Consider an insulated, hollow cylinder with cross sectional area A and height H that is placed upright on a level surface so that the base is completely sealed. The top of the cylinder is open to the atmosphere, which is assumed to be described by an ideal diatomic gas at a temperature T_0 and pressure P_0. You place an insulating disk of mass m and area A (assume it has vanishing thickness) at the top of the vessel. The disk forms an airtight seal and can slide up and down the cylinder without friction, but your hand is initially holding the disk without movement. Throughout this problem gravity is taken to act in the vertical direction (see Fig. 1). A table of equations that may be relevant in the problem is provided below.

(i)



(ii)

Equation	Application
$PV = Nk_BT$	Ideal gas law
$PV^\gamma = \text{const.}$	Isentropic process
$Q = NC_V\Delta T$	Isochoric process
$Q = NC_p\Delta T$	Isobaric process

Figure 1: (i) The cylinder of Question 3. The direction of gravitational acceleration, g, is indicated. (Left) Initially the disk is held in place at the top of the cylinder, forming an airtight seal. (Right) After the disk is lowered (a) or dropped (b)-(c), it moves a distance y. (ii) Some thermodynamic equations that may be relevant.

(a) Assume that you *slowly* lower the disk until it is perfectly kept in place by the pressure exerted by the gas confined in the cylinder. Obtain expressions for the final pressure P_f and temperature T_f of the confined air.

   i) the final pressure P_f and temperature T_f of the confined air,

   ii) the final distance y travelled by the disk from the top of the cylinder.

   Your expressions should be given in terms of the initial temperature and pressure of the gas, T_0 and P_0, as well as other relevant parameters. (3 points) .

(b) Instead of slowly lowering the disk, we could simply let it drop from its initial stationary position at the top of the cylinder and compress the gas. Discuss the physical principle (e.g., a conservation law etc.) you would use to calculate the distance y corresponding to the maximal compression of the confined gas (at which point the disk will be instantaneously at rest)? *You are not required to carry out the calculation to determine y.* (2 points)

(c) Assume that we do drop the disk from its initial stationary position at the top of the cylinder. In an idealized world, the disk would oscillate up and down forever as it compresses the gas and then rebounds. In reality, we know that some energy will be lost to the surrounding atmosphere and some transferred into the gas inside the cylinder. Assume that once the disk has come to complete rest exactly half of the energy has been transferred into the gas inside the cylinder. Compute the final distance y the disk has compressed the gas. (3 points)

(d) Give a physical argument explaining why you expect the distance y obtained in part (c) should be larger or smaller than (a). *You do not need to obtain or explicitly compare the expressions to answer this question!* (2 points)

Question 4: Consider a one-dimensional chain of N localized sites. Each of these sites is occupied by a single polymer that can be in one of three configurations: It can be "straight" with energy 0 or it can "bend" to the left *or* right with energy $\epsilon > 0$ (see Fig. 2).

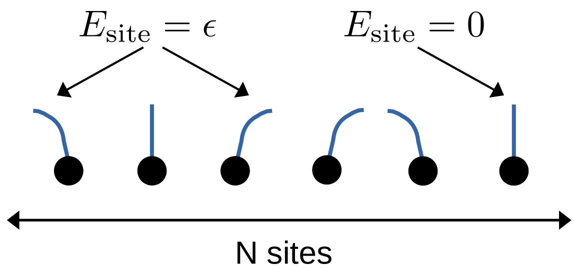


Figure 2: A chain of polymers in 1D that can either be orientated straight upright with energy 0 or bent (to the left or right) with energy ϵ.

(a) By computing the total number of microstates as a function of total energy E and number of sites N, show that the entropy is given by,

$$S(E,N) = Nk_B\left[\frac{E}{N\epsilon}\log(2) - \frac{E}{N\epsilon}\log\left(\frac{E}{N\epsilon}\right) - \left(1-\frac{E}{N\epsilon}\right)\log\left(1-\frac{E}{N\epsilon}\right)\right].$$

To obtain this expression you should assume $N \gg 1$. (3 points)

(b) Show that the total average internal energy as a function of temperature and site number is,

$$E(T,N) = \frac{2N\epsilon}{e^{\epsilon/(k_BT)}+2}.$$

Examine your expression in the limit of low and high temperature and discuss the physical meaning of your results. (3.5 points)

(c) Compute the heat capacity of the chain. Sketch your result as a function of temperature and comment on its behaviour. (3.5 points)

Question 5: Consider a gas of N non-interacting spin-1 bosons in a three dimensional box of volume V. In an applied magnetic field $\mathbf{B} = B\hat{z}$, the single particle energy of each boson is assumed to be given by,

$$E_{\mathbf{p},S_z} = \frac{\hat{\mathbf{p}}^2}{2m} - \mu_0 B S_z,$$

where $\mu_0 = e\hbar/mc$ with m the mass of each boson and $S_z = -1, 0, +1$ is the spin-projection along $\hat{z}$.

In this problem you will find the following integral useful:

$$g_n(y) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{y^{-1}e^x - 1}\, dx.$$

For small z it can be shown that $g_n(z) \approx z + z^2/2^n + \mathcal{O}(z^3)$.

(a) Show that the total number of particles in each of the spin states, i.e., N_{-1}, N_0 and N_{+1}, is given by,

$$N_{S_z} = \frac{V}{\lambda^3} g_{3/2}(z e^{\mu_0 B S_z/k_B T}),$$

where $\lambda = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$ is the thermal de Broglie wavelength. Hint: You will find it useful to know that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. (2 points) .

(b) Write down the expression for the magnetization $M = \mu_0(N_{+1} - N_{-1})$. Use this to show that the zero-field susceptibility is given by,

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{2\mu_0^2 V}{k_B T \lambda^3} g_{1/2}(z).$$

Hint: You will need to use that $y\frac{dg_m(y)}{dy} = g_{m-1}(y)$. (3 points)

(c) Show that the susceptibility given in part (b) collapses to Curie's law in the classical limit,

$$\chi = \frac{2\mu_0^2 N}{3 k_B T}.$$

Justify any approximations you make. (3 points)

(d) In this system a Bose-Einstein condensate (BEC) forms below a critical temperature T_c. Identify which single-particle state becomes macroscopically occupied by the BEC and give an expression for the chemical potential that is valid for $T < T_c$. *You should not need to compute T_c in your answer.* (2 points)