Classical Mechanics and Statistical/Thermodynamics

August 2022

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\,\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$
$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$
$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$

$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$

$\mu_0 = 4\pi \times 10^{-7}$T m/A

electronic charge $e = 1.60 \times 10^{-19}$C

electronic mass $m_e = 9.11 \times 10^{-31}$kg

Density of pure water: 1.00gm/cm^3.

Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K

Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$

speed of light: $c = 3.00 \times 10^8$m/s

Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

Classical Mechanics

1. A small cube with a mass m slides on a frictionless circular surface of radius R that is cut into a large block of mass M as shown in the figure. The horizontal surface that the large block slides on is also frictionless. The system is subject to a constant gravitational force in the $+y$ direction with a gravitational acceleration of g

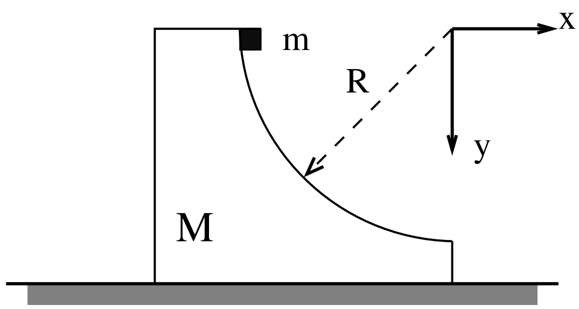

Figure 1: The curved block and the small cube slide without friction.

   (a) Write the total energy of the system in terms of the lab frame with the origin fixed at the position as shown in the picture. (1 point)

   (b) What is the velocity of the small block as it leaves the large block at $y = R$? (1 point)

   (c) What is the velocity of the small block in the limit where $M \gg m$ as it leaves the large block? Explain and interpret your result. (1 point)

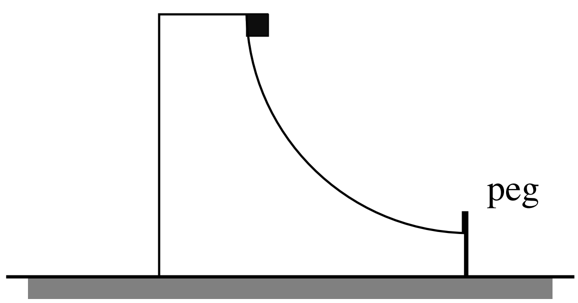

Figure 2: A peg is added to the end of the curved arc of the block.

   (d) A small peg is attached at the end of the curved arc so that the cube collides with it exactly at the bottom of the arc. Describe in words (without calculations) the subsequent motion of the small cube and block if:

      i. The collision is totally inelastic. (1 point)

      ii. The collision is totally elastic. (1 point)

   (e) Redo parts (a) through (d) if the small block is replaced by a small solid sphere with radius r that can roll down the whole circular path without slipping. Assume that $r/R \ll 1$. The moment of inertia of a solid sphere is $\frac{2}{5}mr^2$. (5 points)

2. Two uniform rods of mass, m and length, L are attached with a hinge (a flexible connection). The hinge is held a height above the ground such that each rod makes an angle of $\theta = 30°$ with respect to the ground. The hinge is released from rest and the system falls. The hinge is massless and frictionless, and the ground is frictionless.

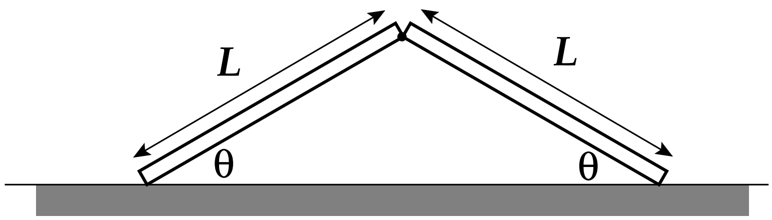

Figure 3: The initial angle $\theta = 30°$.

(a) Find the Lagrangian for this system in terms of θ. (3 points)

(b) Find the Euler-Lagrange equation. (2 points)

(c) Find an expression for $\dot{\theta}$ as a function of θ. (2 points)

(d) Find the speed of the hinge when it hits the ground. (1 point)

(e) Derive an integral expression for the time it takes to hit the floor. You do not have to evaluate the integral. (2 points)

3. Consider a particle of mass m in three dimensions (3D) that is subject only to a force

$$\vec{F}(\vec{r}) = f(r)\,\hat{r}$$

where $r^2 = x^2 + y^2 + z^2$ and $\hat{r} \equiv \vec{r}/r$ is the unit vector pointing in the radial direction.

(a) Show that any motion of the particle will be restricted to a plane.

(b) Construct a Lagrangian for the system and, by identifying conserved quantuities, show that the equation of motion can be reduced to a one dimensional (1D) problem.

(c) Assume the force is characterized by the specific function

$$f(r) = -\frac{\alpha}{r^2} - \frac{\beta}{r^3}$$

where α and β are both positive constants. Without solving the problem explicitly, characterize the possible motion of the particle (e.g. bound, unbound, or otherwise) as a function of β, the initial orbital angular momentum ℓ_0 and the initial energy E_0. Under what conditions does a stable circular orbit exist?

(d) Assume that the particle is initially started in a stable, circular orbit of radius R, as discussed in part (c) and then weakly perturbed to a radius $r = R + \epsilon$ where $\epsilon/R << 1$. Use the action-angle formalism to obtain the period τ of the subsequent small radial oscillations.

Statistical Mechanics

4. It is found that for a particular rubber band with rest length L_0, its tension $f(x,T)$ depends on its length x and temperature T via the formula:

$$f(x,T) = \alpha\, T \left(\frac{x}{L_0} - \frac{L_0^2}{x^2} \right)$$

where $\alpha > 0$. Furthermore its heat capacity at constant length, $c_x(x,T)$ is a constant value, K.

(a) Show $\left.\frac{\partial U}{\partial x}\right|_T = 0$, where $U = U(x,T)$ is the internal energy. This shows that the internal energy is independent of x. (1 point)

(b) Assuming that we know that the value $U(L_0, T_0) = U_0$, find an expression for $U(x,T)$ at arbitrary x and T. (2 points)

(c) Use the above information and a Maxwell relation to calculate $\left.\frac{\partial c_x}{\partial x}\right|_T$. (3 points)

(d) From the previous steps, derive an expression for $S(x,T)$, the entropy as a function of length and temperature. (3 points)

5. A statistical system consists of N non-interacting particles with spin 1/2. The particles are fixed in their position and each possesses a magnetic moment μ. The particles are immersed in a magnetic field B. The Hamiltonian of the system is $H = -\mu B \sum_{i=1}^{N} \sigma_i$, where $\sigma_i = \pm 1$.

   (a) Let $\epsilon = \mu B$ and let $N_\pm$ be the number of particles with $\sigma_i = \pm 1$. Using these definitions, write the energy of the system as a function of N and N_+. Hint: The energy of the system, E, is equal to the Hamiltonian, H, here. Write N_+ and N_- as functions of N and E. (2 points)

   (b) Find the entropy $S(E, B, N)$ of the system. Hint: Find the number of configurations to construct the entropy. Use the Stirling approximation for factorials to simplify your expression, and eliminate N_+ or N_- by replacing them with functions of N and E. (2 points)

   (c) Find the energy $E(T, B, N)$ and, from it, derive the specific heat $C(T, B, N)$ as functions of temperature T, magnetic field B, and total particle number N. (3 points)

   (d) Find the magnetization $M(T, B, N)$ and, from it, the susceptibility $\chi(T, B, N)$. Use your expression for χ to prove the Curie law, i.e. that χ is inversely proportional to the temperature when B goes to zero. (3 points)

6. Chemical potentials and quantum statistical mechanics.

   (a) What is the physical meaning of chemical potential, and how does a system respond to gradients in its value across the system? (1 point)

   (b) In a classical free (but perhaps not ideal) gas, is the chemical potential typically positive, negative or zero, and why? (1 point)

   (c) In a free Fermi gas, what do the limits of positive and negative chemical potential mean, and how do they affect the behavior of the system? (2 points)

   (d) In a free Bose gas, is the chemical potential typically positive, negative or zero, and why? How does this relate to the phenomenon of Bose-Einstein condensation? (2 points)

   (e) A collection of N non-interacting, free spin-$\frac{1}{2}$ particles each of mass m are confined to a two dimensional plane of area $A = L \times L$. The energy of a single particle is given by

$$E(s_z, \vec{k}) = \hbar\, v_0\, k$$

   where $k = \left|\vec{k}\right|$, and $v_0 > 0$. What is the Fermi energy of the system? (2 points)

   (f) Assume instead that the particles above are spin-0 bosons. Determine if they will form a Bose-Einstein condensate at low temperatures. (2 points)