Classical Mechanics and Statistical/Thermodynamics

August 2020

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant $K = 8.998 \times 10^9$ N-m$^2/C^2$

$\mu_0 = 4\pi \times 10^{-7}$T m/A

electronic mass $m_e = 9.11 \times 10^{-31}$kg

Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K

speed of light: $c = 3.00 \times 10^8$m/s

$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$

electronic charge $e = 1.60 \times 10^{-19}$C

Density of pure water: 1.00gm/cm^3.

Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$

Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

Classical Mechanics

1. A frictionless pulley with mass M and radius R in the shape of a uniform disk is hanging from the ceiling and has a heavy uniform rope with linear mass density λ and length L hanging from it. Initially, the rope is at rest with the difference in height between the ends of the rope on the left and right side equal to αL.

   The system is released and one side of the rope falls while the other rises due to the force of gravity, and the pulley rotates so that the rope moves without slipping.

   Assume no slipping of the rope, and that we only consider motion during the period before the left end of the rope reaches the pulley. Also work in the limit that $L >> R$ so that all of the rope is either on the left side or right of the pulley and you neglect the small fraction in contact with the pulley.

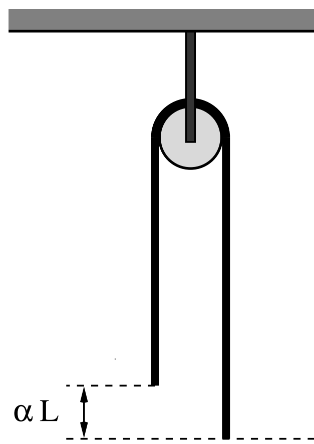

Figure 1: The initial height difference of the two ends of the rope is αL.

   (a) List and explain what mechanical quantity or quantities are conserved in the rope+pulley system. (1 point)

   (b) If initially $\alpha = 1/8$, what is the angular velocity of the pulley when $\alpha y = L/4$? (4 points)

   (c) Now assume that at $t = 0$, the length of each side of the rope is approximately equal, but just different enough to cause the rope to fall to one side ($\alpha << 1$). Determine the angular momentum of the system as a function of time as long as the neither end of the rope is above the center of the pulley. (5 points)

   Be sure to define your co-ordinate system variables clearly.

2. A single particle of mass, m slides without friction on a surface defined by the paraboloid,

$$z = \frac{x^2 + y^2}{\ell}$$

in the presence of a gravitational field $\vec{g} = -g\hat{z}$, where ℓ is a characteristic length that determines the shape of the paraboloid. Define the cylindrical coordinate radius $\rho \equiv \sqrt{x^2 + y^2}$, and azimuthal angle $\phi \equiv \arctan y/x$.

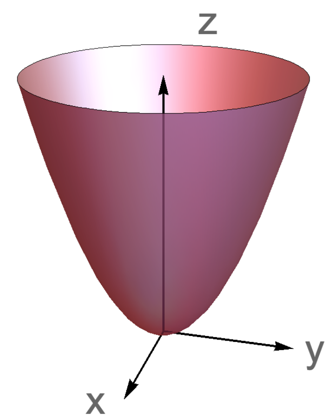

Figure 2: The paraboloid surface $z = (x^2 + y^2)/\ell = \rho^2/\ell$.

(a) Write the Lagrangian in terms of the coordinates (ρ,ϕ). (2 points)

(b) Find the Euler-Lagrange equations for this Lagrangian. (2 points)

(c) What quantities are conserved? Show or explain why. (2 points)

(d) Find the conditions for the radius, ρ and the angular velocity, ω such that the particle will move in a circle. (1 point)

(e) Assume the system is moving in a circle with angular momentum, L. In the co-rotating frame, find the effective potential, and the radius of the circle in terms of L, m, ℓ and g. (1 point)

(f) While moving in this circle, the particle is given a small, radial perturbation. Find the angular frequency of the small, radial oscillations about the value of the radius found in part (e). (2 points)

3. Consider the mottion of a 1D particle with mass m in the Morse potential,

$$V(x) = V_0 \left(e^{-2\alpha x} - 2e^{-\alpha x}\right)$$

where $V_0 > 0$ and $\alpha > 0$.

(a) Find the minimum possible value for the total mechanical energy. (2 points)

(b) Given the total mechanical energy E, determine the turning points of the system. (3 points)

(c) Sketch $V(x)$. (2 points)

(d) Sketch the phase portrait, *i. e.* show $\dot{x}$ as a function of x for a number of different energies, indicating the direction of the paths you draw, and indicating which are higher and lower energies. (3 points)

Depending on whether you prefer a mathematical or graphical approach, you might find it easier to do the parts of the problem in a different order than the one given. You may also find it useful to work with the coordinate $z = e^{-\alpha x}$.

Statistical Mechanics

4. Black body radiation fills a spherical cavity of volume V and radius R, at temperature T, and pressure P. The internal energy U of this photon gas as a function of T and V is

$$U(T, V) = \frac{4\sigma}{c} V T^4$$

where σ is the StefanBoltzmann constant and c is the speed of light.

   (a) Find the constant volume heat capacity c_v for the photon gas. (1 point)

   (b) Find the the internal pressure P for the photon gas. (2 points)

   (c) Find the differential equation for an adiabatic expansion of the photon gas relating T and V . (3 points)

   (d) Find the relationship between T and V along an adiabat. (2 points)

   (e) Assume that the cosmic microwave background (CMB) radiation was decoupled from matter when both were at 3000 K. Currently, the temperature of the CMB radiation is 2:7 K. What was the radius of the spherical universe at the moment of decoupling, compared to now? Consider the process of expansion as adiabatic. (2 points)

5. Consider a system of N distinguishable particles called "fictons." The ground state of a ficton has spin zero ($\sigma = 0$) and zero energy. The first excited state of a ficton has energy ϵ and spin one ($\sigma = 1$), with quantized values of the z-component $\sigma_z \in \{-1, 0, 1\}$. If a magnetic field $\vec{B}$ is applied in the z direction then the excited state energies will shift by an amount $g\mu_0\sigma_z B_z$.

   (a) What is the partition function for the system in the canonical ensemble? (2 points)

   (b) What is the average energy of the system? (2 points)

   (c) If $B_z = 0$, does the fact that the excited state has $\sigma = 1$ have any effect on the average energy as a function of temperature? Why or why not? (1 point)

   (d) What is the magnetic susceptibility of the system? (3 points)

   (e) Discuss the magnetic susceptibility in the limits $\epsilon >> g\mu_0 B_z$ and $\epsilon << g\mu_0 B_z$. How do they differ? (2 points)

6. *The Ideal Bose gas* Consider an ideal gas of noninteracting bosons with spin-0 in uniform space (no spatially varying potentials). Parts (a)-(c) deal with the general case, and parts (d) and (e) with the two-dimensional case.

(a) According to Bose-Einstein statistics, the average occupation number $< N_\epsilon >$ of a quantum state as a function of energy ϵ, chemical potential μ, and temperature T is

$$\langle N_\epsilon \rangle = \frac{1}{\exp\left(\frac{\epsilon-\mu}{k_B T}\right) - 1}, \tag{1}$$

where k_B is Boltzmann's constant. According to this expression, can the chemical potential μ be positive? Why or why not? (1 point)

(b) Write the total number of particles as a sum over the occupation numbers over all energies, using the fugacity $z = e^{\beta\mu}$, with $\beta = \frac{1}{k_B\, T}$. (1 point)

(c) Use $1/(1-x) = \sum_{\ell=0}^{\infty} x^\ell$ and collect terms to show *explicitly* that

$$N = \sum_\ell z^\ell \sum_\epsilon e^{-\ell\beta\epsilon}. \tag{2}$$

(2 points)

(d) Consider this problem now in 2D space. Convert the sum over energies in the expression for N from above into an integral over phase space volume in 2D. Evaluate the integral assuming a large arbitrary area $\int dxdy = S$. Simplify the result to a sum over ℓ, using the deBroglie wavelength $\lambda_{dB} = \sqrt{\frac{2\pi\hbar^2}{mk_bT}}$ (4 points)

(e) Evaluate the remaining sum over ℓ to arrive at a closed-form expression for N, the total number of particles, as a function of S, λ_{dB}, μ and T. Does N diverge when the chemical potential μ approaches zero? Does Bose-Einstein condensation therefore take place in uniform 2D space or not? Why? (2 points)