Classical Mechanics and Statistical/Thermodynamics

August 2018

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\; e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\; e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad\qquad \sum_{n=1}^\infty (-1)^{p+1}\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad\qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad\qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad\qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad\qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

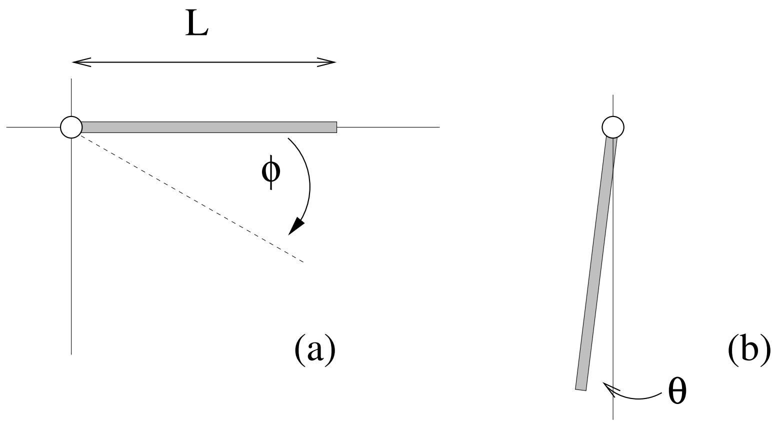
$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad\qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A	electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg	Atomic mass unit: 1.66×10^{-27}kg.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K	Planck's constant: $\hbar = 1.054 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$
speed of light: $c = 3.00 \times 10^8$m/s	Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$
Gravitaional Constant: $6.67 \times 10^{-11}\mathrm{J}\cdot\mathrm{m}\cdot\mathrm{kg}^{-2}$	

Classical Mechanics

1. Two rotating rod problems:



   (a) One end of a negligibly thin rod of length L, mass M and uniform density is attached to an ideal, frictionless pivot. The rod is initially at rest and horizontal. When it is released, it rotates downward, making an angle ϕ with respect to the horizontal.

      i. Calculate the moment of inertia of the rod with respect to the pivot point attached at the end of the rod. (1 point)

      ii. Determine the force exerted by the pivot on the rod as a function of the angle ϕ, between the time of release and the point where $\phi = \pi/2$. Express your answer as the components along the rod direction and perpendicular to that direction. *Note:* You are not asked for the force as a function of time. (3 points).

   (b) A clock has been built using this rod as a physical pendulum to keep time. It is suspended vertically from the frictionless piviot, and makes small oscillations about this point. You may find it easier to work this problem in terms of θ, the angle the rod makes with the vertical axis, so that $\theta = \phi - \pi/2$.

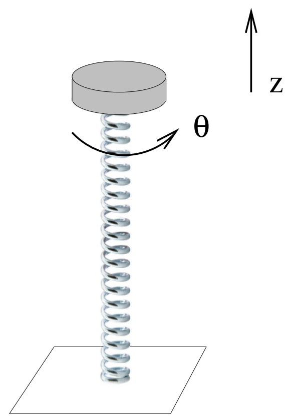
      i. Assume that the pivot is attached to the very end of the rod. Calculate the period of the pendulum oscillations for small amplitude oscillations. (2 points)

      ii. The clock is found to run too fast by a factor of $(1 + \epsilon)$, so that it is gaining one minute every hour. To fix this the pivot is moved down the length of the rod by a distance αL. Derive an exact expression relating α and ϵ in the small angle approximation. (2 points)

      iii. Expand your relation to lowest order in the correction, and determine the numerical value of α. (2 points)

   All of your answers to part (b) should be in terms of the variables g, L, α, M, and ϵ.

2. A torsion spring of spring constant k sits vertically on a table. (A torsion spring is a device that if rotated an amount θ will provide a restoring torque of $\tau = k\theta$.) The spring is massless, but the rotating top is a uniform disk of mass m and radius r. The disk moves up and down as the spring coils and uncoils, so that motion in the z and θ directions are proportional. The disk will move a vertical distance d for every radian it rotates and *vice versa*. At $t = 0$ the disk is motionless and just touching the spring so that $\theta = 0$. Assume that gravity points in the negative z-direction.



(a) What is the kinetic energy of the system, in terms of the rotational velocity $\dot{\theta}(t)$? (1 point)

(b) Derive the potential energy of the torsion spring. (1 point)

(c) Find the Lagrangian in terms of the generalized coordinate θ. (2 points)

(d) Find the Euler-Lagrange equation. (1 point)

(e) Find the equilibrium value of θ of the system if the disk is released and eventually (at long times due to drag) comes to rest. (1 point)

(f) Find the frequency of oscillation of θ about this point. (2 points)

(g) The disk is released at $t = 0$. Find $\theta(t)$ as well as the vertical position as a function of time. (2 points)

3. Hamilton-Jacobi formalism

   A particle of mass m moves in a vertical xz plane under gravity g near the surface of Earth, where z points up (away from the center of the Earth) and $x \perp z$.

   (a) Write the time independent Hamilton-Jacobi equation for this system. (2 points)

   (b) Solve the the Hamilton-Jacobi equation and calculate the Jacobi complete integral $S(x, z, E, t)$, with x, z the coordinates along the horizontal and vertical axes and E the energy. (4 points)

   (c) Using your solution of $S(x, z, E, t)$, calculate x and z and the momenta p_x and p_z as a function of time. (3 points)

   (d) Identify all time-independent constants of the motion. Express each one in terms of generalized coordinates and momenta only. (1 point)

Statistical Mechanics

4. According to Bekenstein and Hawking, the entropy of a black hole is proportional to its surface area A, and given by

$$S(A) = \frac{k_B c^3}{4G\hbar} A$$

where A is the surface area of a spherical black hole of radius R, and the radius of a black hole of mass M is the distance that would make its escape velocity equal to the speed of light,

$$R(M) = \frac{2G}{c^2} M$$

Thus we may express the entropy of a black hole in terms of its mass or its surface area, since their relationship is fixed.

(a) Show that when two black holes of equal mass M_0 combine into a single black hole, the entropy of the system increases and determine by what factor it increases. (2 points)

(b) The internal energy of a black hole is simply given by the Einstein relation:

$$E = Mc^2$$

Use this and the expression for the entropy to find the temperature of a black hole as a function of its mass. (3 points)

(c) A black hole will emit radiation according to the Stefan-Boltzmann law:

$$\frac{dE}{dt} = -\sigma A T^4$$

where σ is the Stefan-Boltzmann constant,

$$\sigma \equiv \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$

where k_B is Boltzmann's constant. This energy loss causes the black hole to slowly evaporate. From the above, determine $M(t)$ for a black hole of initial mass M_0. (3 points)

(d) Derive an expression for the time it takes for a black hole of initial mass M_0 to evaporate. (1 point)

(e) Evaluate your expression to find out how long it would take a black hole the mass of the Earth (6.00×10^{24}kg) to evaporate. (1 point)

5. Consider a simple model for a rubber band consisting of N polymer links each of length ℓ. These links can be in one of six states: they can be point in the $\pm x$-direction, *or* along the $\pm y$-direction *or* along the $\pm z$ direction. The energy of the i-th link is

$$E_i = \begin{cases} -f\ell & \text{if in the } +x \text{ direction} \\ +f\ell & \text{if in the } -x \text{ direction} \\ 0 & \text{otherwise} \end{cases}$$

where f is an externally applied force in the $+x$ direction.

(a) Calculate the partition function for a single link in the canonical ensemble, $Z(T, f, N = 1)$. (2 points)

(b) Calculate the corresponding free energy for a polymer of N links, $G(T, v, N)$. (2 points)

(c) Calculate the average length in the x-direction of a chain made from N links, at temperature T, and under external force f, or $L(T, f, N)$. (2 points)

(d) Show that the thermal expansion coefficient at constant f,

$$\alpha_f \equiv \frac{1}{L}\frac{\partial L}{\partial T}\bigg|_{f,N} = \frac{\partial}{\partial T}\ln L\bigg|_{f,N}$$

is negative and derive an expression for its value. (2 points)

(e) Explain, using the concepts of thermodynamics and statistical mechanics, why α_f should be negative. (2 points)

6. Consider a gas of N free, non-interacting spin-1 bosons with mass m in a two dimensional $L \times L$ system with area $A = L^2$.

   (a) What is the grand canonical free energy $\Xi(T, A, \mu)$? (2 points)

   (b) What is the pressure exerted by the gas? (1 point)

   (c) What is the average energy in the gas? (2 points)

   (d) What is the average number density of particles in the system, N/A? (3 points).

   (e) Does the system exhibit Bose-Einstein condensation at low temperatures? If so, calculate the transition temperature T_c. If not, prove that it does not. (2 points)