Classical Mechanics and Statistical/Thermodynamics

August 2015

Possibly Useful Information

Handy Integrals:

$$\begin{aligned}
\int_0^\infty x^n e^{-\alpha x}\,dx &= \frac{n!}{\alpha^{n+1}} \\
\int_0^\infty e^{-\alpha x^2}\,dx &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \\
\int_0^\infty x\ e^{-\alpha x^2}\,dx &= \frac{1}{2\alpha} \\
\int_0^\infty x^2\ e^{-\alpha x^2}\,dx &= \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \\
\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx &= \sqrt{\frac{\pi}{b}}e^{-a^2/4b}
\end{aligned}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\mathrm{Li}_p(z) \equiv \sum_{n=1}^\infty \frac{z^n}{n^p}$$

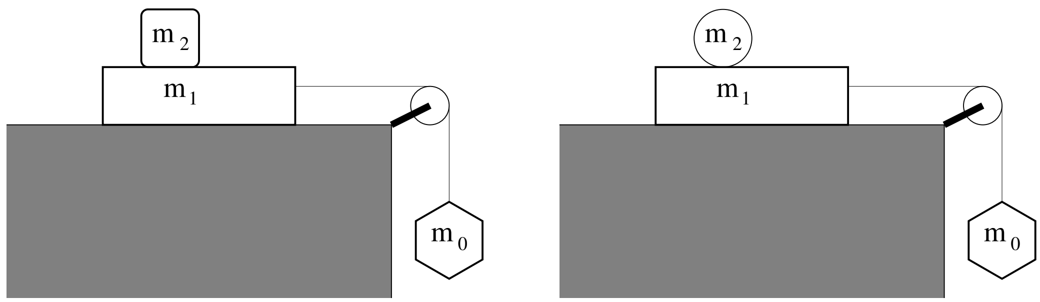
$$\mathrm{Li}_p(1) = \zeta(p)$$

Physical constants:

$$\begin{aligned}
\hbar &= 1.05457\times 10^{-34}\mathrm{m}^2\mathrm{kgs}^{-1} \\
m_{\text{electron}} &= 9.109\times 10^{-31}\mathrm{kg} \\
k_B &= 1.38\times 10^{-23}\mathrm{m}^2\mathrm{kg}\cdot\mathrm{s}^{-2}\mathrm{K}^{-1}
\end{aligned}$$

Classical Mechanics

1. A block of mass m_1 sits on a frictionless horizontal table and is attahed to a hanging block with mass m_0 by an ideal massless rope draped over an ideal massless, frictionless pulley. We wish to put an object on top of the block, and will consider two cases.



   (a) Consider the case on the left, where we place a block of mass m_2 on top of the first block, and release the system from rest. Determine the minimum value of the coefficient of static friction, μ, between blocks 1 and 2 that will prevent block 2 to from slipping when block 1 accelerates to the right. (3 points)

   (b) Consider the case on the right, where we place a sphere of mass m_2 on top of the first block. Assume that the sphere at all times rolls without slipping.

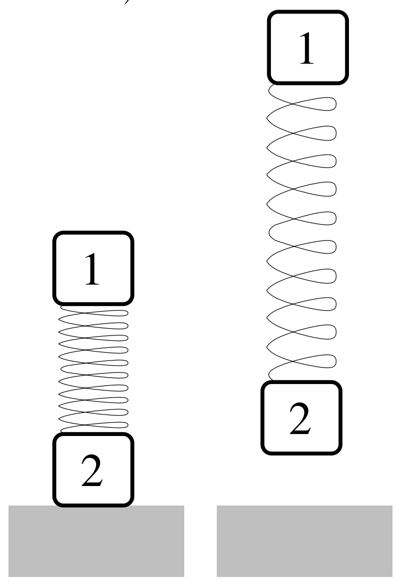
      i. Find the acceleraion of block 1 to the right. (5 points)

      ii. Find the acceleration of the sphere to the right. (1 point)

      iii. Find the rotational acceleration of the sphere. (1 point)

   The moment of inertia of a sphere and mass m and radius R is $(2/5)mR^2$.

2. Consider two identical blocks of mass m connected by an ideal massless spring of spring constant k and equilibrium length ℓ. (The equilibrium length is the length of the spring when there are no net external forces on it, such as when it is decoupled from the blocks) You should assume that $k\ell \gg mg$.



(a) The system is oriented vertically and the top block (labeled block "1") is depressed a distance y_0 and then released from rest. Find $y_0^{(\text{min})}$, the minimum value of y_0 such that the lower block (block "2") is barely lifted off the ground by block 1. Be sure to draw a co-ordinate system and indicate what direction is positive. (3 points)

(b) Now assume that block 1 is compressed an initial distance $y_0 = \ell/2 > y_0^{(\text{min})}$. Determine the trajectories $y_1(t)$ and $y_2(t)$ up until the time the second block hits the ground. You do not have to determine this final time. You will want to look at the trajectories over two different intervals:

i. Before block 2 leaves the ground. (2 points)

ii. After block 2 leaves the ground. (5 points)

Again, be sure to draw a co-ordinate system and indicate what direction is positive.

Assume that all motion is in the vertical direction, and note that you do not have to solve part (a) to solve part (b).

3. Assume that we have generalized co-ordinates $\mathbf{q} = (q_1, q_2)$ and associated momenta $\mathbf{p} = (p_1, p_2)$ that satisfy Hamilton's equations of motion, so that $\dot{q}_i = \{q_i, H\}$ and $\dot{p}_i = \{p_i, H\}$, where the notation $\{x, y\}$ gives the Poisson bracket of x and y with respect to the variables $\mathbf{q}$ and $\mathbf{p}$. We wish to make a transformation to a new set of variables, $\mathbf{q}'(\mathbf{q}, \mathbf{p})$ and $\mathbf{p}'(\mathbf{q}, \mathbf{p})$.

   (a) Show that if

   $$\begin{aligned} \left\{q'_i, p'_j\right\} &= \delta_{i,j} \\ \left\{q'_i, q'_j\right\} &= 0 \\ \left\{p'_i, p'_j\right\} &= 0 \end{aligned}$$

   then

   $$\{F, G\} = \{F, G\}'$$

   for any quantities F and G where $\{x, y\}'$ is the Poisson bracket of x and y with respect to q' and p'. (4 points)

   (b) Consider the system described by the Hamiltonian:

   $$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \cos\left(2q_1 + q_2\right)$$

   We would like to move to a new set of co-ordinates:

   $$\begin{aligned} q'_1 &= 2q_1 + q_2 \\ q'_2 &= q_2 \end{aligned}$$

   which decouples the position co-ordinates. Find the p'_1 and p'_2 that make this transformation canonical, and express the new Hamiltonian in these new co-ordinates. (4 points)

   (c) Looking at the new Hamiltonian, it should be clear that there are two constants of the motion. One will be the Hamiltonian itself, because there is no explicit time dependence. What is the second constant of the motion, in terms of the original co-ordinates? (2 points)

Statistical Mechanics

4. Consider a thermally insulated vessel, divided into two parts by a partition. One side contains n_1 moles of nitrogen gas that occupies a volume V_1 at temperature T_1 and pressure P_1 and the other contains n_2 moles of argon gas that occupies a volume V_2 at T_2 and P_2. Assume nitrogen to be an ideal gas with $c_v = (5/2)R$ and argon to be an ideal gas with $c_v = (3/2)R$. The goal of this problem is to calculate the change in entropy of the system when the partition is removed and each gas expands freely through the container.

   Since entropy is a function of state, the change in entropy between an initial and final state of a system is independent of the path taken to get from one state to another. That means we can break this problem into separate segments of a path connecting the initial and final states such that the entropy change for each segment is more easily calculated.

   (a) First let the two parts of the system equilibrate thermally at constant volumes. Find the final temperature, T_f, and the entropy change of the system. (3 points)

   (b) Second let the pressure of the two parts of the system equilibrate at this constant temperature (i.e., letting the partition between the chambers move). Find the entropy change of the system for this step. (3 points)

   (c) Finally, remove the partition and let the molecules of the gas mix. Find the entropy change for this step. (3 points)

   (d) What is the total entropy change in this process? (1 point)

5. Consider a set of N distinguishable atoms that has an energy given by:

$$E = \sum_{i=1}^{N} \epsilon\, \sigma_i^2 + h\, \sigma_i$$

where $\sigma_i \in \{-1, 0, 1\}$. The quantity σ_i is the value of an atomic spin, ϵ is an internal crystal field, and h is an externally applied magnetic field. We will analyze this system in the canonical ensemble.

(a) What is the partition function for this system, $\mathcal{Z}(T, N)$? (1 point)

(b) What is the internal energy, $U(T, N)$? (2 points)

(c) Calculate the magnetic susceptibility,

$$\chi = \frac{\partial M}{\partial h}$$

where $M \equiv \langle \sum \sigma_i \rangle$ is the magnetization. (4 points)

(d) Show that $\chi(h)$ for large positive ϵ has a peak as a function of increasing βh, and explain physically why this is so. (3 points)

6. A *Dirac fermion* is a particle which obeys Fermi statistics and has an energy given by

$$E(\vec{k}) = \hbar v_0 \, |\vec{k}| = \hbar v_0 \, k$$

where v_0 is a characteristic velocity. In this problem we will work in the grand canonical ensemble and analyze Dirac fermions in a two dimensional system, similar to what is found in graphene.

   (a) Calculate the density of states, $D(E)$ for spin-1/2 Dirac fermions in two dimensions. (3 points)

   (b) What is the Fermi energy, E_f, as a function of the fermion density, N/A, where A is the area of the system? (3 points)

   (c) Calculate the energy of the system as a function of T, A, and μ. (By this we mean that the *thermodynamic variable dependence* is on T, A, and μ. Your answer will involve other mathematical and physical constants such as $\hbar$ or v_0.) Your answer should involve the *Polylogarithm* function, $\mathrm{Li}_p(z)$, defined on page 2. (4 points)