# Classical Mechanics and Statistical/Thermodynamics

**August 2011**

# Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\; e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\; e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

Moments of Inertia:

$$I_{\text{hoop}} = MR^2$$
$$I_{\text{disk}} = \tfrac{1}{2}MR^2$$
$$I_{\text{sphericalshell}} = \tfrac{2}{3}MR^2$$
$$I_{\text{ball}} = \tfrac{2}{5}MR^2$$

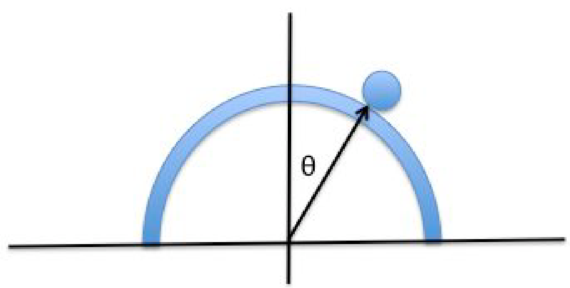
# Classical Mechanics

1. A solid uniform marble with mass $m$ and radius $r$ starts from rest on top of a hemisphere with radius $R$. It will start to roll to the right, and eventually fly off the hemisphere.

   (a) Assume that the marble rolls without slipping at all times. Calculate $\theta_1$, the angle with respect to the vertical at which the marble loses contact with the hemisphere. (3pts).

   (b) Where will the marble hit the ground, as measured from the center of the hemisphere? You may use the variable $\theta_1$ in your answer. (If you do not solve part (a), you can still attempt this problem by writing your answer in terms of this variable.) (3pts).

   (c) Now assume that the force of friction between marble and the hemisphere is $\mu N$, where $N$ is the normal force between the marble and the hemisphere. Calculate the angle $\theta_2$ at which the marble will no longer roll without slipping. (4pts).



2. Consider a point particle of mass $m$ moving under the influence of a central force:

$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\,\hat{r}$$

where $n$ is an integer greater than one ($n = 2, 3, \ldots$), the variable $r$ is the distance from the origin of the force ($r \equiv |\vec{r}|$) and $\hat{r}$ is a unit vector in the radial direction. In this problem, we will examine when circular orbits are stable for such a central force.

   (a) Calculate potential energy of this force. Choose the zero of the potential to be at infinity ($r = \infty$). (1pt)

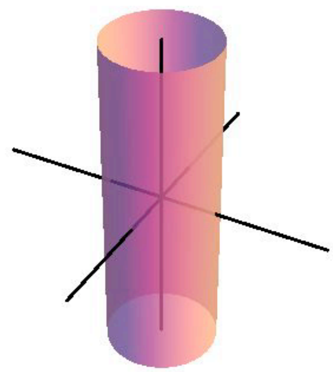
   (b) Show that the angular momentum about the origin, $L$, is conserved. (You may use the Newtonian, Lagrangian, or Hamiltonian formulations of the problem). (2pts)

   (c) Write an expression for the total energy of the particle $E$ as a function of $r$, $dr/dt$, $L$, $k$, and $n$. (1pt)

   (d) Assume the particle is moving in a circular orbit about the origin, so that $dr/dt = 0$. Calculate the radius of the orbit and the velocity of the particle as a function of the above variables. (3pts)

   (e) When is this circular orbit stable? (Hint: look at $dE/dr$ and $d^2E/dr^2$.) (3pts)

3. A particle of mass $m$ is constrained to move on an infinitely long cylinder of radius $a$. The center of the cylinder is oriented along the z-axis, as shown. An attractive central potential, $U(r) = U(\sqrt{a^2 + z^2})$, is located at the origin, where $r$ is the radius is spherical coordinates.



(a) Write down the Lagrangian for the problem. (1pt)

(b) From the Lagrangian, explicitly derive the Hamiltonian for the particle. (2pts)

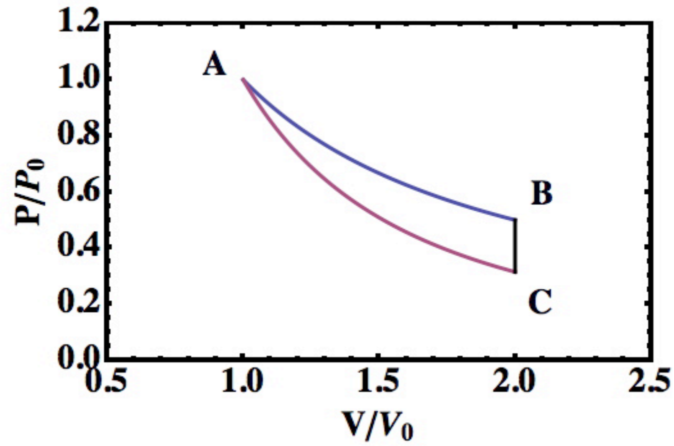
(c) Is angular momentum about the z-axis conserved? Prove your answer. (2pts)

(d) Under what conditions is motion in the z-direction bounded? (2pts)

(e) Assume that the potential is $U(r) = \frac{1}{2}\alpha r^2$. Solve the equations of motion, and reduce the problem to quadrature. (3pts)

# Statistical Mechanics

4. Consider an ideal monatomic gas used as the working fluid in a thermodynamic cycle. The number of particles is $n_0$. It follows a cycle consisting of one adiabat, one isochore and one isotherm, as shown below.




(a) Calculate the pressure, temperature, and volume at each corner of the cycle, A, B, and C, expressing your answer in terms of $P_0$, $V_0$, $n_0$ and perhaps $R$, the ideal gas constant. Note that point A the pressure is $P_0$ and the volume is $V_0$. (3pts)

(b) Calculate the work done on the system, the heat into the system and the change in the internal energy of the system for each process step. (4.5pts)

(c) What direction around the ycle must the system follow to be used as a functional heat engine? (1/2pt)

(d) What is the efficiency of the cycle, run as an engine? (1pt)

(e) What is the efficiency of an ideal Carnot engine run between reservoirs B and C? (1pt)

5. Consider the quantum mechanical linear rotator. It has energy levels

$$E_J = \frac{\hbar^2}{2I} J\,(J+1)$$

where $I$ is the moment of inertia and $J$ is the angular momentum quantum number, $J = 0, 1, 2, \ldots$. Each energy level is $(2J+1)$-fold degenerate.

(a) In the low temperature limit ($\hbar^2/2I \gg kT$) determine approximate expressions for:

i. The rotation partition function. (2pts)
ii. The internal energy. (1pt)
iii. The specific heat. (1pt)

(b) In the high temperature limit ($\hbar^2/2I \ll kT$) determine approximate expressions for:

i. The rotation partition function. (2pt)
ii. The internal energy. (1pt)
iii. The specific heat. (1pt)

(c) How do the quantum results compare with the equipartition theorem for a classical rotator with two transverse degrees of freedom? (2pts)

6. Consider the "bogon," a spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = cp^3.$$

where $p \equiv |\vec{p}|$ Assume that your bogons are confined in a three dimensional sample and are non-ineracting.

   (a) Working in the grand canonical ensemble, determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential, $\mu$ (or the fugacity, $z \equiv e^{\beta\mu}$), $T$, and $V$. (3pts)

   (b) What is the bogonic Fermi energy ($\mu$ at $T = 0$) as a function of their density? (3pts) (*Hint:* This should not involve any complicated integrals).

   (c) Derive a series expansion in $z$ for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. (4pts)