Quantum Mechanics
Qualifying Exam–August 2011

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work**.

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_2^2(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{2i\varphi}$$

$$Y_2^1(\theta,\varphi) = -\frac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{i\varphi}$$

$$Y_1^1(\theta,\varphi) = -\frac{3}{\sqrt{8\pi}}\sin\theta\, e^{i\varphi} \qquad Y_2^0(\theta,\varphi) = \frac{5}{\sqrt{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_1^0(\theta,\varphi) = \frac{3}{\sqrt{4\pi}}\cos\theta \qquad Y_2^{-1}(\theta,\varphi) = \frac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{-i\varphi}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{3}{\sqrt{8\pi}}\sin\theta\, e^{-i\varphi} \qquad Y_2^{-2}(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{-2i\varphi}$$

In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

PROBLEM 1: Postulates of Quantum Mechanics

A physical system consists of three distinct physical states. For this system, an operator Λ has eigenvalues λ_1, λ_2 and λ_3.

(a) Write down the completeness relation for this system. [2 points]

(b) Apply the completeness relation, then write down the expansion of a general state $|\psi\rangle$ in terms of eigenvectors of Λ [1 point]

(c) What is the probability that a measurement Λ of the state $|\psi\rangle$ yields the value λ_1? [2 points]

(d) A measurement of Λ on the state $|\psi\rangle$ is found to give a value λ_2. What is the state of the system immediately after the measurement? [1 point]

(e) A second measurement of Λ on the system is immediately performed. What is the probability of finding $\langle\Lambda\rangle = \lambda_1$? What is the probability of finding $\langle\Lambda\rangle = \lambda_2$? [2 points]

(f) Let us assume that the Hamiltonian H is time independent. Write down an equation that determines the time evolution of the state $|\psi(t)\rangle$ in the Schrödinger picture. Write down an equation that determines the time evolution of $\Lambda(t)$ in the Heisenberg picture. [2 points]

PROBLEM 2: Harmonic Oscillator

A particle of mass m is confined to one dimension. Its potential energy is

$$V(x) = \frac{1}{2} m\,\omega^2 x^2,$$

where $\omega > 0$ is a real parameter. At time $t = 0$, the state of the particle is represented by the real wave function

$$\Psi(x,0) = \frac{1}{\sqrt{2}} \left(1 - \frac{x}{|x|}\right) \phi(x),$$

where $\phi(x)$ is a normalized function of odd parity.

On each question, to receive *any* credit you must fully justify your answer.

(a) At $t = 0$, what is value of the *position probability density* $\mathcal{P}(x,0)$ at the origin, $x = 0$? [2 points]

(b) **Describe** the *parity* of the wave function at $t = 0$ and at any $t > 0$. [2 points]

(c) The *region probability* $\mathcal{P}\big(\,[a,b],t\big)$ denotes the probability that a position measurement at time t would detect the particle in the finite region $x \in [a,b]$. What are the *initial values* of this quantity for the left and right halves of the x axis: $\mathcal{P}\big(\,(-\infty,0\,],0\big)$ and $\mathcal{P}\big(\,[0,\infty),t\big)$? [2 points]

(d) At what time $t_{\text{right}} > 0$, *if any*, is $\mathcal{P}\big(\,[0,\infty),t_{\text{right}}\big) = 1$? [1 point]

(e) At what time $t_{\text{left}} > 0$, *if any*, is $\mathcal{P}\big(\,(-\infty,0\,],t_{\text{left}}\big) = 1$? [1 point]

(f) At what time $t_{\text{same}} > 0$, *if any*, are the two region probabilities equal: $\mathcal{P}\big(\,(-\infty,0\,],t_{\text{same}}\big) = \mathcal{P}\big(\,[0,\infty),t_{\text{same}}\big)$? [2 points]

PROBLEM 3: Angular Momentum Operators

Consider a state space formed from the direct sum of the two subspaces: $\mathcal{E}$(j=0) spanned by $|j=0, m_y=0\rangle$ and $\mathcal{E}$(j=1) spanned by $|j=1, m_y=1\rangle, |j=1, m_y=0\rangle$, and $|j=1, m_y=-1\rangle$,: i.e.

$$\mathcal{E} = \mathcal{E}(\mathrm{j}=1) \oplus \mathcal{E}(\mathrm{j}=0)$$

where

$$J^2|j, m_y\rangle = j(j+1)\hbar^2|j, m_y\rangle$$
$$J_y|j, m_y\rangle = m_y\hbar|j, m_y\rangle$$

.

Let

$$|\Psi\rangle = \frac{1}{\sqrt{5}}|j=1, m_y=1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|j=1, m_y=0\rangle - \frac{1}{\sqrt{2}}|j=0, m_y=0\rangle$$

.

(a) Consider the measurement of the two observables J^2 and J_y. Do these observables commute? Demonstrate explicitly the value of the commutator of J^2 and J_y. **(2 points]**

(b) Determine the probability of measuring J^2 and getting $2\hbar^2$, i.e. determine $P_{|\Psi\rangle}(2\hbar^2 \text{ for } \mathrm{J}^2)$. What is the resulting normalized state vector, $|\Psi'\rangle$ after this measurement? **(2 points]**

(c) If J_y is then measured after the measurement in part (b), what is the probability of obtaining $m_y = 0$, i.e. what is $P_{|\Psi'\rangle}(0 \text{ for } \mathrm{J_y})$? What is the resulting normalized state vector after this measurement? [2 points]

(d) What is the composite probability of measuring J^2 and getting $2\hbar^2$ and then measuring J_y and getting zero, i.e. what is $P_{|\Psi\rangle}(2\hbar^2 \text{ for } \mathrm{J}^2, 0 \text{ for } \mathrm{J_y})$? **(1 point)**

(e) Now starting with the original $|\Psi\rangle$ reverse the measurements, measuring J_y first and getting zero, and then measuring J^2 and getting $2\hbar^2$. Determine four quantities: 1) $P_{|\Psi\rangle}(0 \text{ for } \mathrm{J_y})$; 2) the resulting normalized state $|\Psi''\rangle$; 3) $P_{|\Psi''\rangle}(2\hbar^2 \text{ for } \mathrm{J}^2)$; and 4) the final normalized state after both measurements have been taken. [2 points]

(f) What is the new composite probability when the measurements are reversed, i.e. what is: $P_{|\Psi\rangle}(0 \text{ for } \mathrm{J_y}, 2\hbar^2 \text{ for } \mathrm{J}^2)$? Are your two composite probabilities the same or different? Discuss in detail. [1 point]

PROBLEM 4: Spin Angular Momentum

A Stern-Gerlach experiment is set up with the axis of the inhomogeneous magnetic field in the $x-y$ plane, at an angle θ relative to the $x-$axis. Let us call this direction $\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$. Then the spin operator in the $\hat{r}$ direction is $S_r = \cos\theta S_x + \sin\theta S_y$. Let us describe the common eigenvectors for S^2 and S_i as $|s, m_i\rangle$, e.g. $|s, m_x\rangle$ or $|s, m_z\rangle$.

(a) For a spin$-1/2$ particle, calculate the matrix corresponding to S_r. [1 point]

(b) Evaluate the eigenvalues of S_r. [1 point]

(c) Find the normalized eigenvectors of S_r. [2 points]

(d) Suppose a measurement of the spin of the particle in the $\hat{r}$ direction is made and it is determined that the spin is in the positive $\hat{r}$ direction, i.e. $S_r|\psi\rangle = (+\hbar/2)|\psi\rangle$. Now a second measurement is made to determine m_x (the component of the spin in the x direction). What is the probability that $m_x = -1/2$? [3 points]

(e) Suppose that the particle has spin in the positive $\hat{r}$ direction as in part (d). The z component of the spin is measured and it is discovered that $m_z = +1/2$. Now a third measurement is made to determine m_x. What is the probability that $m_x = -1/2$? [3 points]

PROBLEM 5: Stationary Perturbation Theory

Consider a particle of mass m confined in a 2D infinite square well:

$$V(x,y) = \begin{array}{ll} 0, & \text{for } 0 \le x \le L \text{ and } 0 \le y \le L, \\ \infty, & \text{otherwise,} \end{array}$$

with energy eigenfunctions

$$\psi_{n_x,n_y}(x,y) = \frac{2}{L} \sin\left(\frac{n_x \pi}{L} x\right) \sin\left(\frac{n_y \pi}{L} y\right) .$$

(a) What are the energies and degeneracies of the first four energy levels (eigenenergies) of the particle? Explain your answer. [1 point]

Impurities in the well will shift these energy levels. Assume we can model the effect of an impurity through a local potential:

$$W(x,y) = -V_0 \; L \; \delta(x - x_0)\delta(y - y_0)$$

where the point (x_0, y_0) is the position of the impurity.

(b) For the case where $x_0 = y_0 = L/2$, what are the energy shifts (including splitting of energy levels) to first order in V_0 for the first two energy levels of the particle? Show your work. [3 points]

Which of the energy eigenstates will not be changed by this impurity? Explain. (You should not have to do any calculations to answer this second question.)

(c) Again for $x_0 = y_0 = L/2$, what is the shift in the ground state energy that is second order in V_0? You should write your result in terms of sums, and approximate the result by summing over the largest terms. [3 points]

(d) For the case where $x_0 = L/3$ and $y_0 = L/4$, what are the energy shifts (including splitting of energy levels) to first order in V_0 for the first two energy levels of the particle? Show your work. [3 points]

PROBLEM 6: Variational Method

Consider a Hamiltonian H that may or may not be solved exactly. The variational theorem states that the expectation value of energy obtained from a trial wavefunction will always be greater than or equal to the ground state energy.

Consider a trial wave function ϕ consisting of two basis wavefunctions Ψ_1 and Ψ_2 such that

$$\phi = c_1\Psi_1 + c_2\Psi_2$$

where c_1 and c_2 are constants.

(a) Find the expectation value of the energy for this system. [1 point]

(b) Now assume $\langle\Psi_1|\Psi_2\rangle = \langle\Psi_2|\Psi_1\rangle = 0$, $\langle\Psi_1|H|\Psi_2\rangle = \langle\Psi_2|H|\Psi_1\rangle$ and c_1 and c_2 are real. Determine a 2x2 matrix relationship for the best bound on the energy. [3 points]

(c) Now also assume Ψ_1 and Ψ_2 are orthonormal. Solve the matrix relationship you found in part (b) to determine 2 solutions for the best bound energy. [2 points]

(d) Note that there are 2 solutions to the best bound energy found in part (c). What additional constraint can you apply to remove one of the solutions? [2 points]

(e) Confirm your answer to part (c) by using a Simple Harmonic Oscillator Hamiltonian and setting Ψ_1 to be the ground state eigenfunction and Ψ_2 to be the first excited state eigenfunction of the Simple Harmonic Oscillator [2 points]