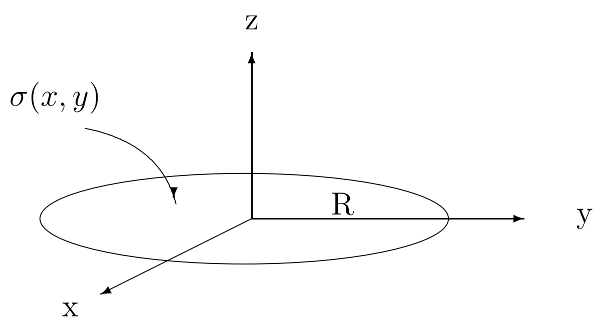
E & M Qualifier

January 14, 2010

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. Consider a thin nonconducting disk of radius R centered on the origin of a coordinate system, lying in the x-y plane, and carrying a surface charge density given by

$$\sigma = \sigma_o \frac{yR}{x^2 + y^2}.$$

   (a) {6 pts} Determine the electric field at a location $\vec{r} = z\hat{k}$.

   (b) {3 pts} Give an approximation to your answer to part (a) that is valid for the $z >> R$.

   (c) {1 pts} Find the force on a charge q located at a position $\vec{r} = z\hat{k}$.

2. Consider a linear, homogeneous, isotropic, and non-dissipative dielectric (i.e., a dielectric where $\mathbf{D} = \epsilon \mathbf{E}$ and ϵ is a constant) in the shape of a sphere of radius R with a point charge Q embedded at its center.

   (a) {2 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$ inside the dielectric.

   (b) {2 pts} Find the bound charge volume density ρ_D inside the dielectric.

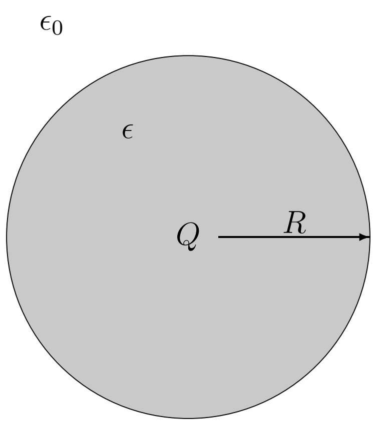
   (c) {1 pts} Find the total bound charge Q_D on the $r = R$ boundary of the dielectric.

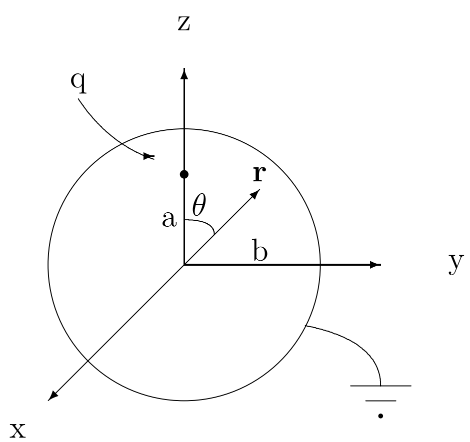
   (d) {2 pts} Find the net charge (free plus bound) at the center of the dielectric.

   (e) {1 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$, outside the dielectric sphere.

   (f) {2 pts} Are $\mathbf{D}$ and $\mathbf{E}$ continuous at $r = R$? If not explain why.

   (If you use Gaussian units you can put $\epsilon_0 = 1$.)





3. A thin grounded hollow conducting sphere of radius 'b' is centered at the origin. A point charge q is located on the z-axis at $z = a < b$ INSIDE the sphere.

   (a) {5 pts} Write the total potential for this system as a sum,

   $$\Phi = \Phi_{sphere} + \Phi_q,$$

   where Φ_q is the potential due to the point charge and Φ_{sphere} (in spherical polar coordinates) is the appropriate linear combination of Legendre polynomials $P_\ell(\cos(\theta))$. Evaluate the coefficients of the $P_\ell(\cos(\theta))$ in the Φ_{sphere} expansion. Recall that the Legendre polynomials are independent orthogonal functions satisfying

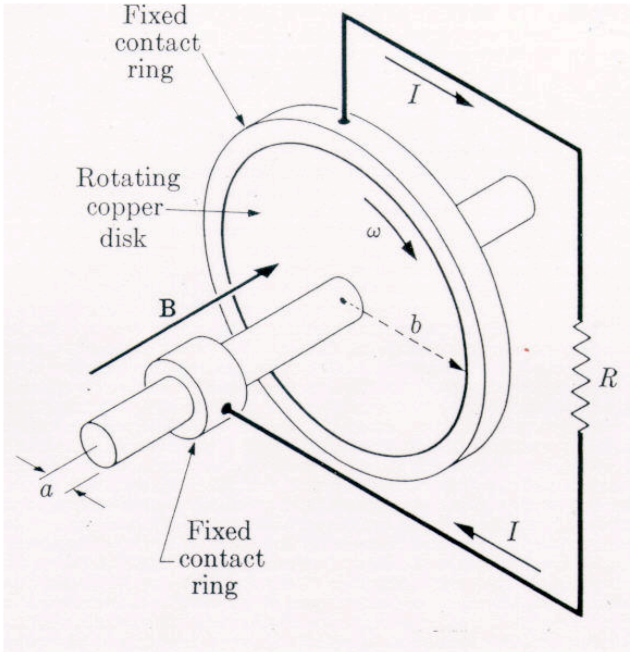
   $$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}$$

   and

   $$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\ell=\infty} \frac{(r_<)^\ell}{(r_>)^{\ell+1}} P_\ell(\cos(\gamma))$$

   where γ is the angle between the two directions $\mathbf{r}$ and $\mathbf{r}'$.

   (b) {5 pts} Show that your expression for Φ_{sphere} is equivalent to the potential of a point charge. Where is the point charge located and what is it's charge?



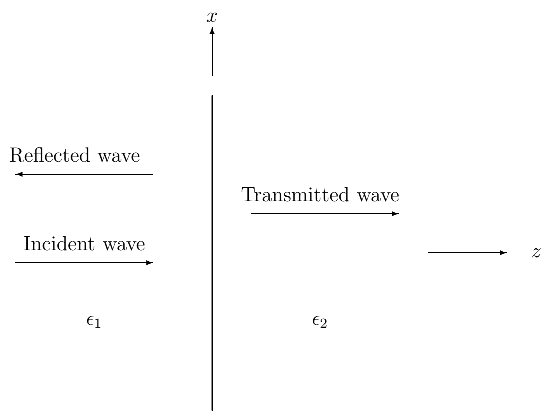
4. The Homopolar Generator consists of a flat copper disk of radius b and thickness t, mounted on an axle of radius a, which mechanically rotates the disk with angular speed ω in the presence of an orthogonal magnetic induction $\mathbf{B}$. A stationary contact ring with inner radius b and negligible resistance surrounds the rotating disk making good electrical and frictionless contact with it. As shown in the figure, the closed electrical circuit consists of the disk and a load resistor R connected by wires between the axle and the stationary contact ring. (Assume the load resistor R is much greater than the resistance of the disk, the contact ring, and the wires.) A constant magnetic induction $\mathbf{B}$ perpendicular to the disk (parallel to the rotation axis) exists between the radii a and b and is zero elsewhere in the circuit.

   (a) {4 pts} Find the current I that flows in the circuit as a function of B, a, b, ω, and R.

   (b) {2 pts}What is the magnitude of the current density $J(r)$ in the rotating disc.

   (c) {2 pts} What torque would you have to apply to the rotating wheel to keep ω from slowing down.

   (d) {2 pts} If σ is the conductivity of copper and t is the thickness of the disk, find the electrical resistance R_d of the disk between the radii a and b. Recall that the resistance of a small length $\Delta\ell$ of conducting material with cross sectional area A is $\Delta R = \Delta\ell/(\sigma A)$.



5. A plane-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by an dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have the same magnetic permeability μ_0 and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0\, e^{i(kz-wt)}\, \hat{x}.$$

   (a) {3 pts} Give the magnetic induction $\mathbf{B}$ associated with the above incoming wave. Make sure your wave satisfies Maxwell's equations, e.g., give k as a function of ω, the direction of $\mathbf{B}$, and the amplitude of $\mathbf{B}$ as a function of E_0.

   (b) {1 pts} Give similar expressions for the $\mathbf{E}$ and $\mathbf{B}$ components of the reflected and transmitted waves. Use E_0'' and E_0' for the respective amplitudes of reflected and transmitted waves.

   (c) {2 pts} In general, what conditions must be satisfied at the junction between two materials by the electromagnetic fields $\mathbf{E}, \mathbf{B}, \mathbf{D}$, and $\mathbf{H}$, if Maxwell's equations are to be satisfied?

   (d) {2 pts}Apply these junction conditions to the combined incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

   (e) {2 pts} Evaluate the time averages of the Poynting vectors of the incident, reflected, and transmitted waves. Recall that

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{1}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

   The sum of the magnitudes of the reflected and transmitted time averaged Poynting vectors should equal the magnitude of the incident wave's time averaged Poynting vector.

6. Maxwell's equations in 4 dimensions

(a) {2 pts} Write the Maxwell equations in the absence of polarizable materials using 4-vector notation, making use of the field strength tensor $F_{\mu\nu}$.

(b) {4 pts} Show that the equations of part (a) reduce to the usual form of Maxwell's equations in 3-vector notation.

(c) {2 pts} The Lagrangian density of the EM field is given by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \qquad\qquad (SI)$$

or

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \qquad\qquad (Gaussian)$$

Recall that all repeated Greek indices are summed over 4-dimensions (1 time and 3 space). Show that the Lagrangian density is invariant under a gauge transformation $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x)$, where α is an arbitrary function of spacetime $x \equiv (ct, \vec{x})$.

(d) {2 pts} If we add an interaction term $\mathcal{L} \rightarrow \mathcal{L} + \Delta\mathcal{L}$ where

$$\Delta\mathcal{L} = j^\mu A_\mu, \qquad\qquad (SI)$$

or

$$\Delta\mathcal{L} = \frac{1}{c} j^\mu A_\mu, \qquad\qquad (Gaussian)$$

to the Lagrangian– where j^μ is some spatially bounded and conserved 4-current density– how does the action $I \equiv \int \mathcal{L} d^4 r$ change under a gauge transformation and do the resulting equations of motion change?