Mechanics and Statistical Mechanics Qualifying Exam
Spring 2013

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\, dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

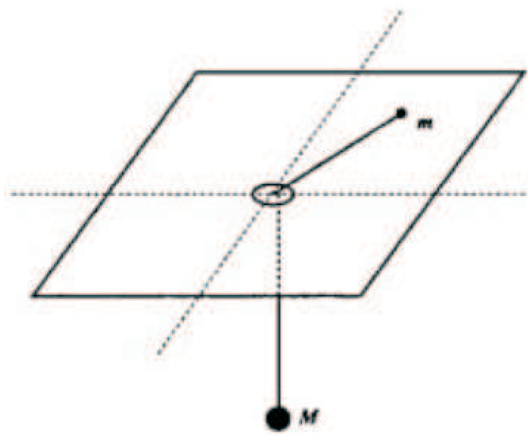
$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = 0.0833333$$
$$\zeta(2) = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = 0.0083333$$
$$\zeta(4) = 1.08232 \qquad \zeta(-4) = 0$$

Problem 1: (10 Points)

A mass m moves on a frictionless table. It is tied to a string that runs through a hole in the table. A mass M hangs from the other end of the string and is acted upon by gravity. M is constrained to move vertically and the hole in table is small and smooth (frictionless).



a. For a mass m orbiting at radius r and velocity v with mass M stationary, determine an equation relating r and v. **(2 Points)**

b. Now imagine replacing the mass M with a force F provided by your hand. What happens if you pull the string to shorten r, what is conserved? How much work, ΔW, is done to change r by Δr? Put your answer in terms of r. **(2 Points)**

c. By pulling the string a distance $d < r$, how does the speed of mass m change? **(2 Points)**

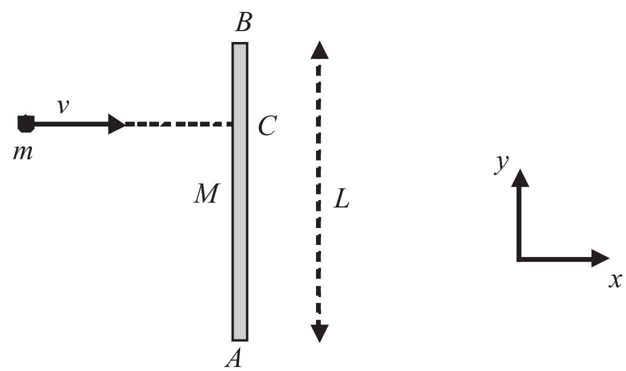
d. Using the expression for ΔW ,in terms of r and Δr from b.), how much work is done to change the orbital radius from r to $r/2$? **(2 Points)**

e. What is the change in angular frequency in part d.)? (Show this for the change from r to $r/2$) **(1 Points)**

f. For the change described in part d.), does the system obey the work energy theorem? **(1 Points)**

Problem 2 (10 Points):

A thin uniform rod of mass M and length $(\bar{AB}) = L$ lies on a horizontal frictionless surface aligned along the y direction as shown below. An object with mass m moving along the x direction with a speed of v collides with the rod at point C.



a. What is the moment of inertia of the rod about point A? **(1 Points)**

b. At what point should the object hit the rod so that immediately after the collision, the rod has pure rotation about the point A? Express your answer for $(\bar{AC})$ in terms of L. **(3 Points)**

c. Now assume the object with mass m collides with the rod at point C such that $(\bar{AC}) = 3L/4$ and the collision is elastic. After the collision, when the rod becomes aligned along the x direction for the first time, what is the distance the center of mass of the rod has moved? For part (c) and forward, assume that $m = M$ (which simplifies the problem) and express your answer in terms of L only. **(4 Points)**

d. At the same moment in time as (c), what is the distance the object with mass m has moved? **(2 Points)**

Problem 3 (10 Points):

A uniform ladder of length ℓ and mass m has one end on a smooth frictionless, horizontal floor and the other end against a smooth, frictionless vertical wall. The ladder is initially at rest making an angle θ_o with respect the horizontal.

a. Using the angle θ (with respect to the horizontal) as the only Lagrangian coordinate, derive an appropriate equation of motion for the time period before the ladder loses contact with the vertical wall. **(2 Points)**

b. Find the height of the upper end of the ladder when the ladder loses contact with the vertical wall. **(2 Points)**

c. Derive a new Lagrangian using the angle θ and the x coordinate of the top of the ladder. **(2 Points)**

d. Find the equations of motion for both coordinates using a Lagrange multiplier. **(2 Points)**

e. What physical quantity in the problem does the Lagrange multiplier represent? **(2 Points)**

f. Repeat part (b) using your new equations of motion. **(2 Points)**

Problem 4 (10 Points):

Consider a rubber band of length L which is being stretched by external force f.

a. Write down the thermodynamic identity (1st law of thermodynamics) relating the change in the internal energy dU to infinitesimal change in length dL, and to the heat TdS. **(2 Points)**

b. In one experiment the length of the band is fixed to $L = 1\,\text{m}$ and the temperature of the band $T = 300\,\text{K}$ is raised by a small amount $\Delta T = 3\,\text{K}$. This causes the force needed to maintain the length of the band to increase by the amount $\Delta f = 1.2\,\text{N}$. In another experiment, the band is stretched from L to $L + \Delta L$ at constant temperature T. As a result, the band exchanges heat with the environment.

1. Find a differential expression for dF, the free energy, in terms of the thermodynamic variables. **(2 Points)**
2. Using your result for the free energy, find the appropriate Maxwell relation for this process. **(2 Points)**

c. What is the amount of heat exchanged with the environment for $\Delta L = 2\,\text{cm}$? **(2 Points)**

d. Is the heat released or absorbed by the rubber band? **(2 Points)**

Problem 5 (10 Points):

A given solid state system consists of N spin 1 atoms, so that the projection of spin on a quantization axis $\sigma \in \{-1, 0, 1\}$. The energy of the i-th atom is

$$E(\sigma_i) = \epsilon \sigma_i^2 + h\sigma_i,$$

where ϵ and h are constants. In this problem you will calculate the partition function in different ensembles.

a. The canonical ensemble: Our goal is to calculate the free energy $F(T, h, N)$.

1. Calculate the partition function $Z(T, h, N)$ in the canonical ensemble. **(1 Points)**
2. From the result in 1., determine the free energy in the canonical ensemble, $F(T, h, N)$. **(1 Points)**
3. What is the magnetization in this ensemble, $M(T, h, N)$? **(2 Points)**

b. The microcanonical ensemble: Our goal is to calculate the entropy S in terms of the extensive quantities, which are the internal energy U, the magnetization M, and the number of atoms, N. Denote the number in each spin orientation as $n_{(-)}$, $n_{(0)}$ and $n_{(+)}$, respectively.

1. Calculate $\Omega(N, n_{(+)}, n_{(-)})$, the number of micro-states available to the system of N atoms for fixed values of $n_{(+)}$ and $n_{(-)}$. **(2 Points)**
2. The total magnetization of the system is given by

   $$M = \mu_0(n_{(+)} - n_{(-)})$$

   and the total internal energy is given by

   $$U(N, n_{(+)}, n_{(-)}) = \epsilon(n_{(+)} + n_{(-)}) + h(n_{(+)} - n_{(-)})$$

   Use these relations and your answer to the question above to determine the entropy in the microcanonical ensemble, $S(U, M, N)$. **(2 Points)**
3. What is the temperature in this ensemble, $T(U, M, N)$ (Hint: use Stirlings approximation)? **(2 Points)**

Problem 6 (10 Points):

A classical system of N distinguishable noninteracting particles each with a mass m is placed in a three-dimensional harmonic well:

$$U(r) = \frac{x^2 + y^2 + z^2}{2V^{2/3}}$$

a. Find the partition function. **(4 Points)**

b. Find the Helmholtz free energy. **(1 Point)**

c. Taking V as an external parameter, find the thermodynamic force $\tilde{P} = -\left(\frac{\partial F}{\partial V}\right)_T$ conjugate to this parameter, exerted by the system. **(1 Points)**

d. Express the equation of state in terms $\tilde{P}, V, T$. **(1 Point)**

e. Find the entropy, internal energy, and total heat capacity at constant volume. **(3 Points - 1 Point for each)**

You may need the following integration formula:

$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{(2n)!}{n! 2^{2n+1}} \sqrt{\frac{\pi}{a^{2n+1}}}$$