Mechanics and Statistical Mechanics Qualifying Exam
Fall 2013

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

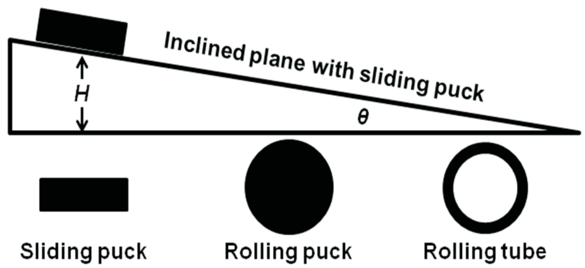
$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$\zeta(1) = \infty$	$\zeta(-1) = 0.0833333$
$\zeta(2) = 1.64493$	$\zeta(-2) = 0$
$\zeta(3) = 1.20206$	$\zeta(-3) = 0.0083333$
$\zeta(4) = 1.08232$	$\zeta(-4) = 0$

Problem 1: (10 Points)

Imagine a race between different objects rolling or sliding down a simple inclined plane. Let the mass of each object be M, the angle of the inclined plane be Θ, and the height be H, where the center of mass of all the objects change by H over a race. You will consider a cylinder with walls of negligible thickness and puck (short solid cylinder) in this problem. Assume the rolling objects roll without slipping.



a. Determine the velocity of a sliding puck (short cylinder) at the bottom of the inclined plane. **(3 Points)**

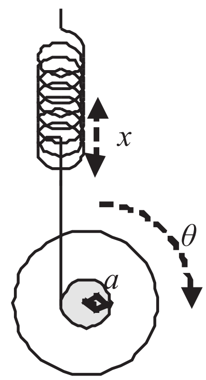
b. Determine the velocity of a round, symmetrical, smooth rolling object whose moment of inertia is $I = \alpha M R^2$, where α is a geometrical factor and R is the radius. **(3 Points)**

c. Show which wins, a sliding or rolling puck? **(2 Points)**

d. Discuss how your result depends on α, M and R. Use your answer to determine if a rolling puck or tube of mass M and negligible thickness would win a race down the incline. **(2 Points)**

Problem 2 (10 Points):

A yo-yo with a mass of m and moment of inertia I falls straight down and spins due to gravity. The string unwinds from the yo-yo around an axle of radius a. The other end of the string is attached to an ideal spring with spring constant k. Define x as the extension of the spring measured with respect to its unstretched length.



a. Using the generalized coordinates x and Θ write the Lagrangian for this system. **(2 Points)**

b. Derive the Lagrange equations of motion. **(2 Points)**

c. Derive a differential equation that describes the oscillation of the spring while the yo-yo is falling down and unwinding. **(2 Points)**

d. What is the oscillation frequency of the spring while the yo-yo is falling down and unwinding? **(2 Points)**

e. Consider the limit of a thin axle ($ma^2 \ll I$) and solve the differential equation found in (c) for the variable x. **(2 Points)**

f. Explain in words the motion described by the equation found in (e). **(1 Points)**

Problem 3 (10 Points):

A spherical pendulum consists of a particle of mass m that is in a gravitational field $\vec{g}$ and is constrained to move on the surface of a sphere of radius ℓ. Use the polar angle θ (measured from the downward vertical) and the azimuthal angle ϕ.

a. Derive the Lagrangian for this system. **(2 Points)**

b. Derive the Hamiltonian for this system. **(2 Points)**

c. Find the Hamiltonian equations of motion. **(1 Points)**

d. Consider the system is undergoing uniform circular motion in ϕ at constant polar angle θ_o. Assuming small variations in θ, expand the Hamiltonian in θ to second order around $\theta = \theta_o$. **(4 Points)**

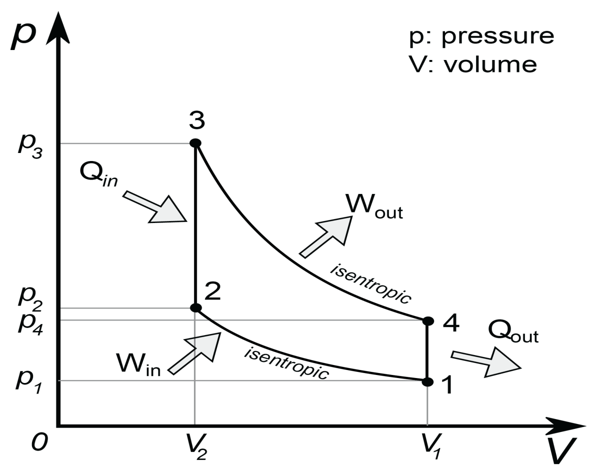
e. Show that the motion in θ is simple harmonic with angular frequency given by:

$$\omega^2 = \frac{g}{\ell \cos\theta_o}(1 + 3\cos^2\theta_o).$$

(1 Points)

Problem 4 (10 Points):

The diesel engine uses the Otto cycle. Below is the P-V diagram for this process. Assume a monatomic ideal gas.



a. Find the work done during each cycle. **(3 Points)**

b. Find the heat exchanged each cycle. **(3 Points)**

c. What is the efficiency of this engine? **(3 Points)**

d. To produce work, which way does the cycle operate? Clockwise or counter clockwise in the diagram. **(1 Points)**

Problem 5 (10 Points):

An electron confined to a 1D ring of radius R in a perpendicular magnetic field B has energy levels

$$\begin{aligned} E(m,\phi) &= \frac{\hbar^2}{2mR^2}\left(m - \frac{\phi}{\phi_0}\right)^2 \\ &= \epsilon\left(m - \frac{\phi}{\phi_0}\right)^2 \end{aligned}$$

where $\phi = \pi R^2 B$ is the magnetic flux through the ring, ϕ_0 is the magnetic flux quantum ($\phi_0 = e/\hbar c$) and m is the angular momentum quantum number, $m = 0, \pm 1, \pm 2, \ldots$. In this problem we will consider a set of N rings, and neglect the spin of the electron.

a. In the high temperature limit ($\epsilon \equiv \frac{\hbar^2}{2mR^2} << kT$) determine approximate expressions for:

1. The canonical partition function, $Z(T, N, B)$. **(1 Point)**
2. The internal energy, $U(T, N, B)$. **(1 Point)**
3. The magnetization, $\mathcal{M} \equiv \frac{\partial U}{\partial B}$. **(2 Points)**

b. In the low temperature ($\frac{\hbar^2}{2mR^2} >> kT$) and weak field ($-\phi_0/2 < \phi < \phi_0/2$) limit determine approximate expressions for:

1. The canonical partition function, $Z(T, N, B)$. **(1 Point)**
2. The internal energy, $U(T, N, B)$. **(1 Point)**
3. The magnetization, $\mathcal{M} \equiv \frac{\partial U}{\partial B}$. (If your result is quite complicated, make sure that you keep only the leading term in part (i) above. **(2 Points)**

c. Are your results similar or different? Explain either why they are similar or why they differ. **(2 Points)**

Problem 6 (10 Points):

Consider a system consisting of a large number N of distinguishable, noninteracting particles. Each particle has only two (nondegenerate) energy levels: 0 and $\epsilon > 0$. Let E/N denote the mean energy per particle in the thermodynamic limit $N \to \infty$.

a. What is the maximum possible value of E/N if the system is not necessarily in thermodynamic equilibrium? **(1 Point)**

b. What is the value of E/N if the system is in equilibrium at temperature T? **(4 Points)**

c. Explicitly take the low ($T \to 0$) and high ($T \to \infty$) limits of your result of part a). Sketch your results. **(2 Points)**

d. Find the entropy per particle $s = S/N$. **(2 Points)**

e. Explicitly take the $T \to 0$ and $T \to \infty$ limits of your result of part d). Explain. **(2 Points)**