

Quantum Mechanics Qualifying Exam - August 2014

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write your alias on the top of every page of your solutions
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \psi + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \psi}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi.$$

One dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \quad P = -i \sqrt{\frac{\hbar m \omega}{2}} (a - a^\dagger)$$

Spherical Harmonics:

$$\begin{aligned} Y_0^0(\theta, \phi) &= \frac{1}{\sqrt{4\pi}} \\ Y_1^0(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos \theta \\ Y_1^{\pm 1}(\theta, \phi) &= \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \\ Y_2^0(\theta, \phi) &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) \\ Y_2^{\pm 1}(\theta, \phi) &= \mp \sqrt{\frac{15}{8\pi}} (\sin \theta \cos \theta) e^{\pm i\phi} \\ Y_2^{\pm 2}(\theta, \phi) &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi} \end{aligned}$$

PROBLEM 1: Stationary and Non-Stationary States

Consider a quantum system whose particles are in the following state:

$$\Psi(x, t) = \frac{1}{\sqrt{8}}\psi_1(x)e^{-iE_1t/\hbar} - i\sqrt{\frac{3}{8}}\psi_3(x)e^{-iE_3t/\hbar} + \frac{1}{\sqrt{2}}\psi_5(x)e^{-iE_5t/\hbar}, \quad (1)$$

where $\psi_n(x)$, $n = 1, 2, 3 \dots$ are stationary states of the Hamiltonian governing the system,

$$H\psi_n(x) = E_n\psi_n(x).$$

Answer the following questions:

- a) Do you expect $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle E \rangle$ to be time dependent or time independent? Discuss briefly, but do not calculate. (2 Points)
- b) Is the uncertainty ΔE positive, negative or zero? Is ΔE time dependent or time independent? Again, discuss briefly but do not calculate. (2 Points)
- c) Is $\Psi(t)$ above a solution of the time dependent Schroedinger equation? Demonstrate. (2 Points)
- d) If the stationary states $\psi_1(x)$, $\psi_3(x)$ and $\psi_5(x)$ are eigenstates of the harmonic oscillator, will any of your answers to part a) change? Justify. (2 Points)
- e) Now assume the particles are in the state

$$\Psi(x, t) = \psi_3(x)e^{-iE_3t/\hbar}.$$

Answer parts a) and b) for this state. (2 Points)

PROBLEM 2: Oscillator Model of Angular Momentum

Arbitrary angular momentum can be constructed from spin-1/2. The latter can be described in terms of the Pauli matrices

$$\mathbf{S} = \frac{\hbar}{2} \boldsymbol{\sigma}.$$

The construction of a general angular momentum can be done by introducing two sets of independent harmonic oscillators, in terms of creation (a_{ζ}^{\dagger}) and annihilation (a_{ζ}) operators,

$$[a_+, a_-] = 0, \quad [a_+^{\dagger}, a_-^{\dagger}] = 0, \quad [a_{\zeta}, a_{\zeta'}^{\dagger}] = \delta_{\zeta, \zeta'},$$

with $\zeta, \zeta' = \pm$ indexing oscillators of type \pm . Now define

$$\mathbf{J} = \frac{\hbar}{2} a^{\dagger} \boldsymbol{\sigma} a,$$

where a is a two component operator,

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.$$

a) Given the form of the Pauli matrices, give the explicit form for J_x, J_y, J_z in terms of a_{ζ}^{\dagger} and a_{ζ} operators (2 Points).

b) Show that $J_{\pm} = J_x \pm iJ_y$ have particularly simple forms in terms of a_{ζ} and a_{ζ}^{\dagger} operators (1 Point).

c) Compute the commutator $[J_x, J_y]$. How is this generalized for the other components? (2 Points)

d) Show that

$$J^2 = J_z^2 + J_+ J_- + i[J_x, J_y],$$

and then write this in terms of the number operators for the two harmonic oscillators,

$$n_+ = a_+^{\dagger} a_+, \quad n_- = a_-^{\dagger} a_-.$$

Show that this implies that the eigenvalues of J^2 are $j(j+1)\hbar^2$, where j is an integer or an integer plus $\frac{1}{2}$ (Hint: apply the J^2 operator in the two harmonic oscillator state $|n_+, n_-\rangle$) (3 Points).

e) Using the properties of the harmonic oscillators, show that the state in which J^2 has the eigenvalue $j(j+1)\hbar^2$ and $J_z = m\hbar$ can be constructed from the state in which both n_+ and n_- have the value zero, $|0\rangle$, by

$$|jm\rangle = \frac{(a_+^{\dagger})^{j+m}}{\sqrt{(j+m)!}} \frac{(a_-^{\dagger})^{j-m}}{\sqrt{(j-m)!}} |0\rangle.$$

(2 Points)

PROBLEM 3: Perturbation Theory

Consider a particle of mass m trapped inside a 1D parabolic potential

$$V(x) = \frac{1}{2}m\omega^2x^2,$$

where ω sets the frequency of oscillation inside the potential.

a) If the particle is perturbed by a *static* potential

$$V_I = \alpha x,$$

with α small, compute energy correction of the energy levels in the lowest order where the result is non-zero. (3 Points)

b) What is the perturbed ket in the ground state? Compute the expectation value $\langle x \rangle$ in this state. Interpret the sign of $\langle x \rangle$. (3 Points)

c) Assume from now on that $\alpha = 0$. Imagine that the particle is charged and sits in the ground state at $t = -\infty$. Suppose an electric field is gradually tuned on, increases to a maximum at $t = 0$ and then slowly dies away,

$$V_I'(t) = -e|\mathbf{E}|x e^{-t^2/\tau^2},$$

where e is the electric charge, and \mathbf{E} is the electric field. Write down the general expression for the amplitude of transition from a generic level i to level f . (Do not solve the integral yet) (2 Points).

d) Evaluate the probability of having the particle in the first excited state at $t = +\infty$. (2 Points).

Hint: $\int_{-\infty}^{\infty} dt e^{-t^2/\tau^2} e^{i\omega t} = \sqrt{\pi\tau} e^{-\omega^2\tau^2/4}$

PROBLEM 4: Two Particles in a 1D Box

Consider two noninteracting particles of mass m inside a 1D box,

$$V(x) = \begin{cases} 0 & , 0 < |x| < a \\ \infty & , \text{otherwise} \end{cases} .$$

Make sure to consider the spin part of the wavefunction in this problem.

- a) Let n_1 and n_2 be the quantum numbers of particle 1 and 2 respectively. What are the wavefunctions of the single particle states for the each particle in the box? What are the single particle energies? (2 Points)
- b) If the particles are distinguishable what is the two-particle wavefunction that describes the state? What is the energy? Write out explicitly the state (or states) and energies for the ground state and first excited states of the system. (2 Points)
- c) If the two particles are identical spin 0 bosons what are the ground state and first excited state wavefunctions and energies? (2 Points)
- d) If the two particles are identical spin 1/2 fermions what are the ground state and first excited state wavefunctions and energies? (2 Points)
- e) Write down the Hamiltonian for the two particles in the box and show that when the particles are identical H commutes with the exchange operator. (2 Points)

PROBLEM 5: Addition of angular momenta

Consider an electron. We know its orbital angular momentum $\ell = 1$ and the z component $m = 1/2$ of its total angular momentum j .

- What are the possible values of j ? (2 Points).
- Write down the kets $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$ in terms of products of spin and orbital angular momentum states (3 Points)
- Calculate the expectation value of the spin operator \mathbf{S} in the state $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. Consider all possible values of j . (3 Points).
- The magnetic dipole moment of the electron is

$$\boldsymbol{\mu} = \frac{e}{2m_e c}(\mathbf{L} + 2\mathbf{S}),$$

with \mathbf{L} the orbital angular momentum operator, e the electron charge, m_e the mass and c the speed of light. Calculate the expectation value of $\boldsymbol{\mu}$ in the states $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. (2 Points)

Raising and lowering angular momentum operators:

$$J_{\pm}|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

PROBLEM 6: Variational approach

A particle with mass, m , moving in one dimension finds itself in a potential given by,

$$V = \infty \quad \text{for } x < 0$$

and

$$V = \beta x^3 \quad \text{for } x > 0$$

where β is a positive constant.

a) Find an approximation to the ground state energy, using the trial wavefunction

$$\Psi = 0 \quad \text{for } x < 0$$

and

$$\Psi = Cxe^{-\alpha x} \quad \text{for } x > 0.$$

where C and α are positive constants. (5 Points)

b) Would you expect the exact ground state energy to be less than your answer to part (a), or greater than it? Justify. (3 Points)

c) How would you go about finding an excited state in this system using the same approach? (2 Points)

Hint: $\int_0^\infty x^2 e^{-ax} = 2a^{-3}$, for $a > 0$.