E & M Qualifier

August 15, 2019

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

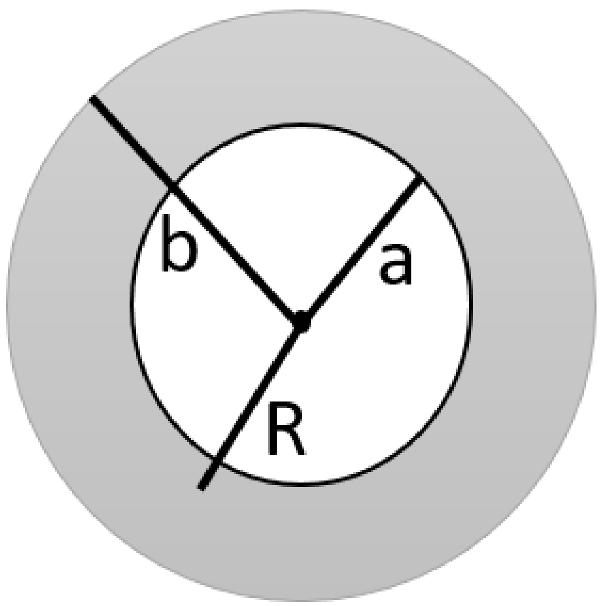
Problem 1: Electrostatics

Consider an infinitely-long line of charge with linear charge density λ.

(a) Use explicit integration of infinitesimal charge elements to determine the magnitude of the electric field a distance x from the line of charge. [4 points]

(b) That line of charge is placed at the center of an infinitely-long hollow cylinder with inner radius a and outer radius b, as shown below. The cylinder carries a uniform charge density ρ. The electric field is zero at a distance R from the line of charge, with $a < R < b$. Determine ρ as a function of λ, a, b, and R. [3 points]

(c) Use your result from part b) to determine the electric field a distance r from the line of charge, with $r > b$. [3 points]



Problem 2: Magnetostatics

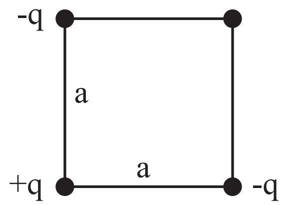
In this problem you will solve for the $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$ fields generated by a sphere of radius R and magnetization $\vec{\mathbf{M}} = M\hat{\mathbf{z}}$.

(a) Starting from the time independent Maxwell equations in matter and assuming that there are no free currents or charges, derive the Poisson equation for a magnetic scalar potential. Identify the source for the potential and the relation of the potential to the appropriate field. [3 points]

(b) Specify the boundary conditions for the appropriate field and the potential and briefly justify them. [2 points]

(c) Using the Poisson equation derived in part (a) and the boundary conditions in part (b) calculate the $\vec{\mathbf{H}}$ and $\vec{\mathbf{B}}$ fields both inside and outside the sphere. [5 points]

Problem 3: Interaction forces and energies



(a) Three charges are situated at the corners of a square as shown above. How much work does it take to bring in another charge, $+q$, from far away and place it in the fourth corner? How much work does it take to assemble the whole configuration of four charges starting at the top left in counter-clockwise order? Does this last result depend on the order in which charges are assembled? Why or why not? [3 points]

(b) Use Gauss' law to show explicitly that the electric field inside a uniformly charged sphere centered at the origin with total charge Q and radius R is $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$, for $r \leq R$. What is the field outside the sphere? [1 point]

(c) Find the net force $\mathbf{F}_{net}$ that the southern hemisphere of a uniformly charged sphere exerts on the northern hemisphere. Express your answer in terms of the radius R and the total charge Q. Compare this to the force between point charges: what distance d would two equal point charges, charge $Q/2$ each, need to have in order for one charge to experience the same force magnitude $\mathbf{F}_{net}$ as the northern hemisphere? Express d in terms of R. Would you have expected your result for d to be larger or smaller than R? Why? Does your result agree with your expectation? [6 points]

Problem 4: Radiation

For a magnetic dipole with an oscillating frequency ω and dipole moment $\vec{m} = m_o \cos(\omega t)\,\hat{z}$, the vector and scalar potentials using the Lorenz gauge are:

$$\vec{A}(\vec{r},t) = \frac{\mu_o m_o}{4\pi}\frac{k}{r}\left[\frac{1}{kr} - i\right]\sin\theta \exp\left(i\left(kr - \omega t\right)\right)\hat{\phi} \quad (1)$$

$$V(\vec{r},t) = 0 \quad (2)$$

when the distance from the dipole (centered at the origin) is large compared to the dipole size and the dipole size is small compared to the wavelength.

(a) Explain the directional dependences of the vector and scalar potentials.
Hint: Recall how the potentials depend on the charge density and current density. [2 points]

(b) Determine the magnetic field in the radiation zone, where $kr \gg 1$. [3 points]

(c) Determine the electric field in the radiation zone, where $kr \gg 1$. [2 points]

(d) Calculate the time-averaged Poynting vector in the radiation zone. [2 points]

(e) Calculate the time-averaged power radiated by the magnetic dipole. [1 point]

Problem 5: Gauges and potentials

(a) Write down Maxwell's equations in covariant form in terms of the field strength tensors $F^{\mu\nu}$ and/or $\tilde{F}^{\mu\nu}$. Use Gaussian units. [2 points]

(b) Re-write the inhomogeneous Maxwell's equations in terms of the four-vector potential A^{μ}. [2 points]

(c) What is the Lorenz gauge expressed in terms of A^{μ} and how does this simplify the inhomogeneous equations? [2 points]

(d) In terms of the $\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ operator, what is the covariant wave equation for which the Greens function $D(x - x')$ is a solution? [2 points]

(e) The retarded Green function $D_r(x - x')$ can be found via Fourier transforming to be $D_r(x - x') = \frac{\Theta(x_0 - x_0')}{4\pi R}\delta(x_0 - x_0' - R)$ (where $R = |\vec{x} - \vec{x}'|$ and $x_0 = ct$). In terms of D_r, write down the solution for A^{μ} as an integral equation involving the Green function D_r and the four-source J^{μ}. Don't forget to include also the solution A^{μ}_{in} to the homogeneous wave equation. [2 points]

Problem 6: Matter and radiation

An electric dipole $\vec{p}$ is oscillating with harmonic time dependence $e^{-i\omega t}$ which generates $\vec{H} = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n} \times \vec{p})$ with $\vec{E} = Z_0 \vec{H} \times \hat{n}$ in the radiation zone ($r \gg \lambda/2\pi$) and where $Z_0 \equiv \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Also, $\hat{n}$ is a unit vector pointing from the dipole to the observation point and $k = 2\pi/\lambda$.

(a) Identify the spherical wave factor in the radiation field. [1 point]

(b) Compute the Poynting vector $\vec{S}$ for radiation into the far zone. [2 points]

(c) Compute the power radiated into solid angle $d\Omega$ ($dP/d\Omega$) in terms of the $\vec{E}$ and $\vec{H}$ fields. [2 points]

(d) Display the dependence of the power radiated $dP/d\Omega$ in terms of the source dipole moment $\vec{p}$. [2 points]

(e) For the dipole oriented along the z-axis, what is the angular dependence of the power radiated? Sketch the pattern of power radiated over all space. [3 points]

Hint: $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$.