Classical Mechanics and Statistical/Thermodynamics

August 2014

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\ dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

Physical constants:

$$\hbar = 1.05457\times 10^{-34}\mathrm{m^2kgs^{-1}}$$

$$m_{\mathrm{electron}} = 9.109\times 10^{-31}\mathrm{kg}$$

$$k_B = 1.38\times 10^{-23}\mathrm{m^2kg\cdot s^{-2}K^{-1}}$$

Classical Mechanics

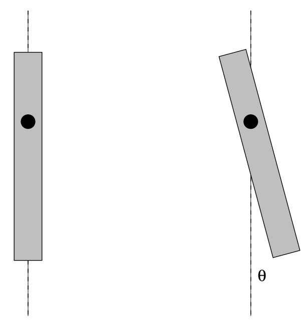
1. A bar of mass M and length L has a hole drilled in it one third along its length. This is used as a frictionless pivot point.

   (a) Calculate the moment of inertia of the system about the pivot point. You may assume that the rod is one-dimensional. (2 points).

   (b) In equilibrium, the bar hangs vertically ($\theta = 0$). The bar is rotated about its pivot, and released from rest when $\theta = \pi/2$. Caculate the angular velocity of the system when $\theta = 0$. (3 points)

   (c) Find the force exerted by the pivot in the above case just as θ passes through zero. (2 points)

   (d) Find an expression for the period of oscillation for small angles $\theta << 1$. Express you answer in terms of the quantities given, and g, the acceleration due to gravity. (3 points).



2. Consider the Earth as a frame rotating about its axis with frequency ω. In this frame of reference particles are subject to the Coriolis force given by:

$$\vec{F}_c = -2m(\vec{\omega} \times \vec{v}_r)$$

where $\vec{v}_r$ is the velocity of the particle relative to the rotating frame and m is the mass of the particle. Choose the z-axis to be normal to the surface of the Earth and pointing outward from the center; the y-axis to point North, and the x-axis to point East. Let $z = 0$ be on the surface of the Earth. Assume you are in the Northern Hemisphere at a colatitude θ (that is, the angle between the z-axis and $\vec{\omega}$ is θ).

The goal of this problem is to calculate the "horizontal" deflection of an object thrown "straight up" in the rotating reference frame of a person on the surface of the Earth.

(a) A particle close to the surface of the Earth ($z << R$ where R is the radius of the Earth) moves under the influence of gravity and the Coriolis force. (You may neglect the centrifugal force in this non-inertial frame for this problem). Write the equations of motion in the x, y, and z directions, under the approximations that $|v_z| >> |v_x|$ and $|v_z| >> |v_y|$. (2 points)

(b) The particle starts from rest (in the rotating frame) and is dropped from a height h above the ground, and it arrives at the ground with a speed $\sqrt{2gh}$. Find the magnitude and direction of the deflection of its end point due to the Coriolis force. (3 points)

(c) The particle is now thrown vertically upward from the ground with an initial speed v_0 so that it reaches a maximum height h and then falls back to the ground. Find the magnitude and direction of the Coriolis deflection. (3 points)

(d) Compare your results from parts (b) and (c) above. Explain why they are the same or different. (1 point).

(e) In the above calculation of the deflection we neglected the centrifugal force; was that reasonable? Why? (1 point)

3. Consider a particle confined to one dimension subject to a force increasing linearly in time, $F = At$.

   (a) Find the Hamiltonian of the system. (2 points)

   (b) What is the corresponding Hamilton-Jacobi equation? (2 points)

   (c) Show that Hamilton's principal function, S can be written in the form:

   $$S = \frac{1}{2}At^2 x + \alpha x - \phi(t),$$

   where α is a constant. (2 points)

   (d) (x pts) Solve the resulting equation for $\phi(t)$, finding the position and momentum as a function of time. (3 points)

   (e) Verify your solution by directly solving Newton's equations of motion for this force and compare this to your answer above. (1 point)

Statistical Mechanics

4. The latent heat of melting ice is L per unit mass. A bucket contains a mixture of water and ice at the melting point of the ice, T_0. We want to use an ideal, maximally efficient, cyclic (reversible) refrigerator to freeze a mass m amount more of the liquid water in the bucket into ice. Assume that this refrigerator is powered by some external source of work, and that it rejects all heat to a **finite** external reservoir of constant heat capacity C and initial temperature T_0.

   (a) What is the change in the entropy of:

      i. The ice and water mixture in the bucket, S_{bucket}, (2 points)

      ii. The external reservoir, S_{res}, (2 points)

      iii. The refrigerator aparatus S_{fridge} itself when run over several cycles, (1 point)

      during the process where a mass m of the water is turned into ice?

   (b) What is the change in the Gibbs free energy during the process? (1 point)

   (c) What is the minimum mechanical work required to run the refrigerator for this process? *Hint:* The most efficient process will have the smallest **total** entropy change. (4 points)

5. A *lattice gas* is a system of volume V divided into N_s cells of volume b so that $N_s = V/b$. Each cell can have either one or no atoms in it ($n_i \in \{0, 1\}$) and has c nearest neighbors. The energy of the system is:

$$E(\{n_i\}) = -\left[\frac{1}{2}\sum_i \sum_{j \in n.n.} w n_i n_j\right]$$

where w is an interaction between adjacent atoms, and the sum over j is restricted to the nearest neighbors of i and the factor of $1/2$ is present to avoid double counting. We will work in the grand canonical ensemble so that the total number of particles,

$$N = \sum_i n_i$$

is not fixed.

(a) Write down an expression for the grand canonical partition function $\mathcal{Z}(T, V, \mu)$ as a sum over the values of n_i. Because this expression depends on the product of n_i and n_j, you will not be able to evaluate it. (1 point)

(b) In the mean field approximation we rewrite the energy as

$$E_{mf}(\{n_i\}) = -\left[\frac{1}{2}\sum_i c w \bar{n}\, n_i\right]$$

where we have replaced n_j by it's average value, $\bar{n}$. This value must be determined self-consistently, so for the moment treat it as simply a constant. Calculate $\mathcal{Z}(T, V, \mu; \bar{n})$, performing the sum over n_i. (2 points)

(c) From your partition function calculate the average value of the number of atoms, $\langle N \rangle = N(T, V, \mu; \bar{n})$. (3 points)

(d) In order for this result to be self-consistent, we must have that

$$\frac{N(T, V, \mu; \bar{n})}{N_s} = \bar{n}$$

which can be thought of as the intersection of the function $N(\bar{n})/N_s$ with the "function" $f(\bar{n}) = \bar{n}$, (that is, a line of slope unity). Explain the behavior of this intersection as μ is varied from large negative to large positive values. (4 points).

6. Consider a photon gas enclosed in a volume V and in equilibrium at a temperature T. The photon is a massless, spinless particle so that $\epsilon(p) = pc = \hbar\omega$, where c is the speed of light. Photons can have two possible transverse polarizations.

   Throughout this problem you may reduce integrals to quadrature, i.e. definite integrals containing no physical parameters and thus equivalent to simple constants.

   (a) What is the chemical potential of the gas? (Recall that the number of photons is *not* conserved). (**1 point**)

   (b) Write down the grand thermodynamic potential Ω and replace the sum with an appropriate integration. (**2 points**)

   (c) Using your result of part a) and b) extract the temperature dependence of the free energy F. (**2 point**)

   (d) Using your result of part c) extract the temperature dependence of the pressure P and the entropy S. (**2 point**)

   (e) Using your previous results, find the relationship between the energy E and the free energy F. (**1 point**)

   (f) Write down the energy as $E = E(P, V)$. (**1 point**)

   (g) Find the temperature dependence of the number of photons N. (**1 point**)