# Classical Mechanics and Statistical/Thermodynamics

**January 2017**

# Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\, dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$
$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$
$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant $\mathrm{K} = 8.998 \times 10^9$ N-m$^2/C^2$

$\mu_0 = 4\pi \times 10^{-7}$T m/A

electronic mass $m_e = 9.11 \times 10^{-31}$kg

Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K

speed of light: $c = 3.00 \times 10^8$m/s

$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$

electronic charge $e = 1.60 \times 10^{-19}$C

Atomic mass unit: $1.66 \times 10^{-27}$kg.

Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2$kg/s

Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

# Classical Mechanics

1. Two rope problems:

   (a) An ideal rope of mass $M$ and length $L$ is suspended at rest so that the bottom of the rope just contacts a scale. The initial force on the scale is zero. At time $t = 0$ the chain is released. Calculate the force on the scale as a function of time as the rope falls and collides with the scale. (3 points)

   (b) The same ideal rope is now placed on a frictionless table, of height $L$. A small fraction of the rope of length $\epsilon L$ initially hangs over the edge. The rope is initially at rest, and is pulled over the edge of the table by the weight of the portion that is over the edge. Calculate the position of the lower edge of the rope as a function of time.

   Make the assumption that the horizontal momentum of the rope sliding on the table is converted into vertical momentum, so that the rope does not fly off sideways from the table. (4 points)

   (c) Calculate the speed of the of the rope as it just hits the floor in the limit that $\epsilon \ll 1$. (3 points)

2. A wedge of mass $M$ and angle $\theta$ moves frictionlessly along the $x$ axis. A small mass, $m$ a distance $\ell$ from the top of the wedge moves frictionlessly along the wedge. Other than gravity and the normal force on the wedge from the ground, there are no external forces on the system.

   (a) (2 pts) F1nd the kinetic energy of the system in terms of the generalized coordinates $x$ and $\ell$.

   (b) (1 pts) Find the Lagrangian.

   (c) (2 pts) Find the equations of motion for the generalized coordinates and the ratio $\mu = m/(m + M)$.

   (d) (1 pt) Is there a speed of the wedge in which the acceleration of the mass is up the wedge? If so, find it. If now, prove mathematically or explain why.

   (e) (2 pts) Integrate the equations of motion and find how long it takes for the particle to slide off if it starts at a height, $h$ above the ground with both wedge and particle at rest.

   (f) (1 pt) Show that in the limit of $M \to \infty$ this agrees with the expected result.

   (g) (1 pt) How far did the wedge move during this time?

3. Consider a free particle of mass $m$ moving on a plane $(x, y)$ in a rectangular infinite square well potential

$$V = \begin{cases} 0 & \text{for } 0 < x < a,\ 0 < y < b \\ \infty & \text{otherwise.} \end{cases}$$

a) Write down the Hamilton-Jacobi equation in cartesian coordinates. (2 points)

b) Find the characteristic function of Hamilton $W$ in terms of the constants of integration. (2 points).

c) Compute the action variables of the problem $J_x$, $J_y$. (3 points).

d) Calculate the frequencies of the motion $\omega_x$, $\omega_y$ and find the condition for periodic trajectories. Interpret your result. (3 points)

# Statistical Mechanics

4. A mono-atomic gas has the fundamental thermodynamic relation:

$$S(U, V, N) = Nk_B \ln \left( \frac{U^{3/2}(V - Nb)}{N^{5/2}} \right)$$

where $U$ is the energy of the system, $N$ is the number of gas molecules, $V$ is the voume of the system and $b$ represents the excluded volume around an individual atom.

(a) Calculate the three equations of state, $P(U, V, N)$, $T(U, V, N)$, and $\mu(U, V, N)$. (3 points).

(b) Calculate the work done during an isothermal expansion of the gas. (2 points)

(c) Calculate the work done during an adiabatic expansion of the gas. (2 points)

(d) Calculate the efficiency of a Carnot cycle using this gas using a hot reservoir at temperature $T_h$ and a cold reservoir at temperature $T_c$. (3 points)

5. An ideal gas of $N$ indistinguishable non-interacting particles is confined in two dimensions and subject to a harmonic trapping potential, so that the energy of the system is:

$$H = \sum_i \frac{\vec{p_i}^{\,2}}{2m} + \frac{1}{2}k\vec{r_i}^{\,2}$$

There is also a hard wall to the 2D system at $r = R_0$.

(a) Calculate the canonical partition function, $Z(T, A, N)$. (2 points)

(b) Calculate the energy, $E(T, A, N)$. (2 points)

(c) Calculate the specific heat. (2 points)

(d) Explain the behavior of the specific heat in the limits:

i. $\beta k R_0^2/2 \gg 1$ (1 point)

ii. $\beta k R_0^2/2 \ll 1$ (1 point)

(e) Calculate the pressure, $P(T, A, N)$, and explain its behavior in the limit $\beta k R_0^2/2 \gg 1$. (2 points)

6. Consider a gas of $N$ free, non-interacting fermions (spin 1/2) in a three a dimensional system with volume $V$.

   (a) What is the fermi energy (the chemical potential at $T = 0$) for the gas as a function of $N$ and $V$? (2 point)

   (b) What is the average energy of the fermi gas, $U(N, V, T = 0)$? (2 points)

   (c) Calculate the pressure in the gas at $T = 0$, as a function of $V$ and $N$. (1 point)

   (d) We will approximate a white dwarf star as a collection of fully ionized carbon-12 atoms, with a density of $10^9\text{kg/m}^3$ and a temperature of $10^6\text{K}$.

      i. If you treat the carbon nuclei as a classical ideal gas, what is the pressure of the gas? (2 points)

      ii. How does this compare to the electron pressure, if we use the $T = 0$ result for electrons above? (2 points)

   (e) Why is it valid to consider the $T = 0$ formula when the actual temperature is $10^6\text{K}$? (1 point)

   You may find it helpful to refer to the table of constants at the front of the exam in evaluating your expressions.