Vectors & Tensors

Einstein notation

Reading Assignment this week Greenberg Chap 13 - 14 - 15 ~~and 16~~

Any repeated index indicates summation

Thus,

$$\underline{a}\cdot\underline{b} = \sum_i a_i b_i = \underbrace{a_i b_i}_{\text{Einstein notation}}$$

Einstein practical guy!!

Examples:

$$|a| = (a_i a_i)^{1/2}$$

$$\bar{x}_j = l_{ij}\, x_i \quad \leftarrow \text{change of coordinates}$$

l_{ij}: cosine of angles between axis i and j.

$\bar{x}_j$: vector component j in new coordinate system

Any entity that behaves like this is defined as a vector

$$\bar{a}\cdot\bar{b} = \bar{a}_j \bar{b}_j = l_{ij}\, a_i\, l_{pj}\, b_p =$$

$$= \underbrace{l_{ij}\, l_{pj}}_{=\delta_{ip}}\, a_i b_p = \delta_{ip}\, a_i b_p = a_i b_i$$

$= \delta_{ip}$ / Axes are orthogonal

$\delta_{ip} b_p = b_i$

$$a \times b = (a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3) \times (b_1 \underline{e}_1 + b_2 \underline{e}_2 + b_3 \underline{e}_3)$$

$$= (a_2 b_3 - a_3 b_2)\underline{e}_1 + (a_3 b_1 - a_1 b_3)\underline{e}_2 + (a_1 b_2 - a_2 b_1)\underline{e}_3$$

$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Let

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any } ijk \text{ are the same} \\ -1 & ijk \text{ is odd permutation of } 123 \\ +1 & ijk \text{ is even perm. of } 123 \end{cases}$$

$$\Rightarrow \quad \boxed{a \times b = \varepsilon_{ijk}\, a_i b_j \underline{e}_k}$$

Similarly the triple scalar vector is:

$$a \cdot (b \times c) = \varepsilon_{ijk}\, a_i b_j c_k \quad (*)$$

Finally the triple vector product is

$$a \times (b \times c) = \varepsilon_{ijk}\, a_j\, \varepsilon_{k\ell m} b_\ell c_m$$

TENSORS

Second order tensors behave like this

$$\bar{A}_{pq} = l_{ip}\, l_{jq}\, A_{ij}$$

and they can be treated as matrices.

Quotient rule: if A_{ij} is a set of nine quantities ~~there~~ b is independent of A and c is a vector and $A_{ij} b_j = c_i \Rightarrow A_{ij}$ is a tensor.

To prove it you need to show that $\left\{ \begin{array}{l} Ab = c \Rightarrow \bar{A}\bar{b} = \bar{c} \\ A \text{ transforms as a tensor} \end{array} \right.$

Gradient

$$\nabla_i = \frac{\partial}{\partial x_i} \Rightarrow \nabla_i \varphi = \frac{\partial \varphi}{\partial x_i}$$

gradient vector operator

gradient of φ

$\nabla \varphi$ is a vector. Indeed $\frac{\partial \varphi}{\partial \bar{x}_j} = \frac{\partial \varphi}{\partial x_i} \frac{\partial x_i}{\partial \bar{x}_j} =$

$$= l_{ij} \frac{\partial \varphi}{\partial x_i}$$

Prove $\nabla \varphi$ is the direction of maximum increase of φ.

$\underline{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i}$. Let $\underline{u}\, dr$ be a small displacement.

$$\Rightarrow \lim_{\delta r \to 0} \frac{\varphi(\underline{x} + \underline{u}\, dr) - \varphi(x)}{dr} = \frac{\partial \varphi}{\partial u}.$$

Using Taylor's Theorem.

$$\frac{\partial \varphi}{\partial u} = \underline{\nabla} \varphi\, \underline{u}$$

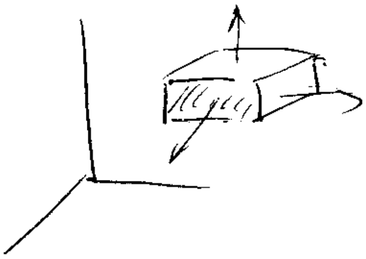
$\Rightarrow$ The maximum change takes place when u and $\underline{\nabla} \varphi$ are colinear.

Divergence

$$\operatorname{div} \underline{a} = (\underline{\nabla} \cdot \underline{a}) = a_{i,i} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$$

watch the notation

Consider a small volume. Let us construct



$$\iint \underline{a} \cdot \underline{n} \, dS$$

If $\underline{n}$ denotes the normal to $dx_2 dx_3$

$$\Rightarrow [a_1(x_1 + dx_1), \xi_2 \xi_3] - a_1(x_1, \xi_2 \xi_3) \, dx_2 dx_3 =$$

$$= \frac{\partial a_1}{\partial x_1} dx_1 dx_2 dx_3 + O(d^4) = \frac{\partial a_1}{\partial x_1} dV + O(d^4)$$

What did I use here?

$$\Rightarrow \iint \underline{a} \cdot \underline{n} \, dS = (\underline{\nabla} \cdot \underline{a}) \, dV$$

$$\lim_{dV \to 0} \frac{1}{dV} \iint \underline{a} \, \underline{n} \, dS = \underline{\nabla} \cdot a$$

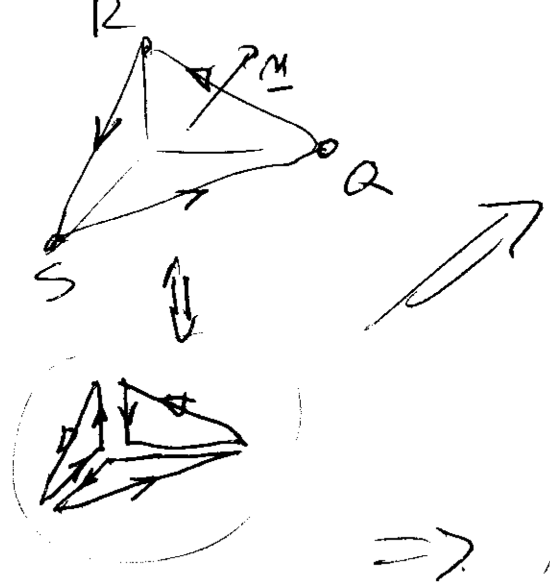
If $\underline{a}$ is thought as a flux $\Rightarrow \iint a \cdot n \, dS$ is the net flux out of the volume $\Rightarrow$ no generation/consumption of the property in the field. (conservation).

$\nabla \underline{a} = 0 \Rightarrow \underline{a}$ is a <u>solenoidal field</u>

$\underline{\nabla}(\underline{\nabla} \varphi) = \nabla^2 \varphi = 0 \Rightarrow \varphi$ is a <u>potential function</u>

Laplacian

In general consider a plane triangle (6)
whose normal is $\underline{n}$



$$\oint_{QRS} \underline{a}\,\underline{t}\,ds = \oint_1 \underline{a}\,t\,ds + \oint_2 \underline{a}\,t\,ds + \oint_3 \underline{a}\,t\,ds$$

$$= \varepsilon_{ijk}\, dA_i\, a_{k,j}$$

$$\Rightarrow \lim_{dA\to 0} \left(\frac{1}{dA} \oint_{QRS} \underline{a}\,\underline{t}\,ds\right] = (\text{curl}\,\underline{a})\,\underline{n}$$

which can be generalized for any curve.
(How?)

$\nabla \times \underline{a} = 0 \quad \Rightarrow \quad \underline{a}$ is irrotational

because evidently any circulation around any infinitesimal curve, vanishes.

Note that if $\underline{a} = \underline{\nabla}\varphi \Rightarrow$

$\underline{\nabla} \times \underline{\nabla}\varphi = 0$, which may make you think that all irrotational fields can be represented by a scalar function

Green - Lagrange - Gauss - Ostrogradsky Theorem
(4) (1) (2) (3)

or **Divergence Theorem.**

$$\int_V \underline{\nabla} \cdot \underline{a} \, dV = \int_S \underline{a} \cdot \underline{n} \, dS$$

which follows directly from our previous findings that for an infinitesimal volume.

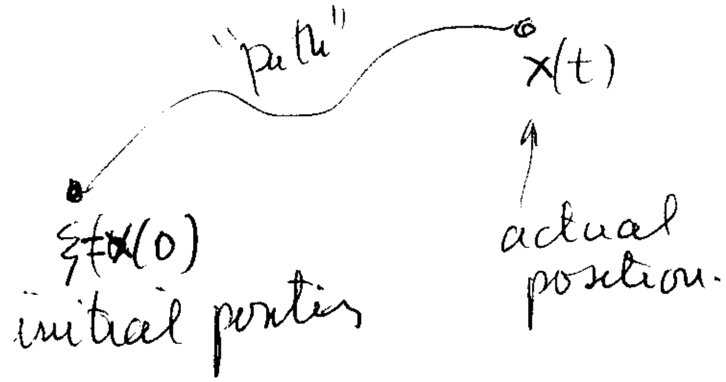
$$\underline{\nabla} \cdot \underline{a} \; dV = \underline{a} \cdot \underline{n} \, dS$$

Stokes - Kelvin · Theorem

$$\oint_C \underline{a} \cdot \underline{t} \, ds = \int_S (\nabla \times a) \cdot \underline{n} \, dS$$

which follows from previous findings for infinitesimal areas.

Kinematics of Motion



$x = x(\xi, t)$ means the path **exists**! (or paths exist)

x : spatial coordinates

ξ : material coordinates (Lagrangian or convected are other names)

Inversion is possible $\Rightarrow$

$\xi = \xi(x, t)$ exists.

which means path is unique! (particles do not break up and split). or (particles cannot occupy two places at the same time).

$$\Rightarrow J = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \neq 0$$

Derivatives

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_x$$

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_{\xi}$$

Thus it makes sense to write the velocity of a particle as

$$\underline{v} = \frac{d\underline{x}}{dt}$$

(when $\frac{\partial v}{\partial t} = 0$ the flow is steady)

In addition, let $F = F(\xi, t)$ be a property.

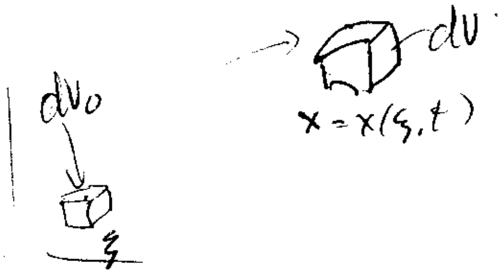
$$\Rightarrow \frac{dF}{dt} = \frac{\partial}{\partial t} F(\xi, t) = \frac{\partial}{\partial t} F(x(\xi, t), t)$$

$$= \frac{\partial F}{\partial x_i} \underbrace{\left(\frac{\partial x_i}{\partial t}\right)_{\xi}}_{v_i} + \left(\frac{\partial F}{\partial t}\right)_x$$

$$\Rightarrow \boxed{\frac{dF}{dt} = \left(\frac{\partial F}{\partial t}\right) + v_i \frac{\partial F}{\partial x_i}}$$

Dilatation

$$dV = \frac{\partial(x_1\, x_2\, x_3)}{\partial(\xi_1\, \xi_2\, \xi_3)}\, d\xi_1\, d\xi_2\, d\xi_3 = J\, dV_0$$



$\Rightarrow$ Now. $J = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ - & - & - \\ \dots & - & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix}$

Let us calculate $\frac{dJ}{dt}$

~~Consider~~ one such term is $\begin{vmatrix} \frac{\partial v_1}{\partial \xi_1} & \frac{\partial v_1}{\partial \xi_2} & \frac{\partial v_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix} = \begin{vmatrix} \left(\frac{\partial v_1}{\partial x_k}\frac{\partial x_k}{\partial \xi_1}\right) & \left(\cdot\ \frac{\cdot}{\partial \xi_2}\right) \\ - & - \\ - & - \end{vmatrix}$

sum of 3 terms each one with the ~~first~~ row derivated.

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial \xi_j}\right) = \frac{\partial}{\partial \xi_j}\frac{dx_i}{dt} = \frac{\partial v_i}{\partial \xi_j}$$

expanding this determinant one obtains $= \frac{\partial v_1}{\partial x_1} J$

$$\Rightarrow \frac{dJ}{dt} = \nabla v \cdot J \Rightarrow \frac{d \ln J}{dt} = \nabla v$$

Which states why $\nabla v = 0$ represents an incompressible fluid.

Reynolds Transport Theorem

Let $F(t) = \iint_{V(t)} \mathcal{F}(x,t)\, dV$

can be scalar vector or tensor

rate of change of $\int \mathcal{F} dV$

$$\frac{d}{dt}\int_{V(t)} \mathcal{F}(\underline{x},t)\,dV = \frac{d}{dt}\int_{V_0} \mathcal{F}[x(\xi,t),t]\, J\, dV_0$$

$$= \int_{V_0}\left(\frac{d\mathcal{F}}{dt} J + \mathcal{F}\frac{dJ}{dt}\right) dV_0$$

what did I do here?

$$= \int_{V_0}\left(\frac{d\mathcal{F}}{dt} + \mathcal{F}\nabla\underline{v}\right) J\, dV_0$$

and here?

$$= \int_{V(t)}\left(\frac{d\mathcal{F}}{dt} + \mathcal{F}\nabla v\right) dV$$

$$= \int_{V(t)}\left\{\frac{\partial\mathcal{F}}{\partial t} + \nabla(\mathcal{F}v)\right\} dV = \int_{V(t)} \frac{\partial \mathcal{F}}{\partial t}\, dV + \int_{S(t)} \mathcal{F}\underline{v}\,\underline{n}\, dS$$

$\int$ of the rate of change at the point

net flow through interface

First consequence:

Conservation of mass and continuity equation

$$m = \iint_V \rho(\underline{x},t)\, dV$$

but $\dfrac{dm}{dt} = 0 \quad \Rightarrow \int_V \left\{ \dfrac{d\rho}{dt} + \rho(\nabla \underline{v}) \right\} dV = 0$

$$\Rightarrow \frac{d\rho}{dt} + \rho \nabla \underline{v} = \frac{\partial \rho}{\partial t} + \nabla(\rho \underline{v}) = 0$$

Continuity equation

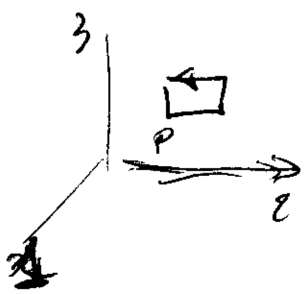
Curl (rot a in older texts) 5

$$\nabla \times \underline{a} = \text{curl } \underline{a} = \varepsilon_{ijk}\, a_{k,j}\, \underline{e}_i$$

Components are

$$\left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right), \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)$$

Let us calculate $\oint \underline{a}\cdot\underline{t}\, ds$ on a rectangle



Consider the vertical sides. The contribution of these sides is

$$a_3(x_1, x_2+dx_2, \xi_3) - a_3(x_1, x_2, \xi_3)\, dx_3$$

$$= \frac{\partial a_3}{\partial x_2} dx_3\, dx_2 + O(d^3)$$

what did I use here?

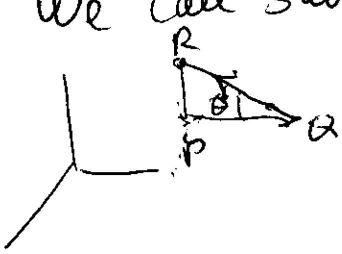
and so on.

$$\oint_1 \underline{a}\cdot\underline{t}\, ds = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) dx_2\, dx_3 + O(d^3)$$

$$\Rightarrow \lim_{dA_1 \to 0} \frac{1}{dA_1} \oint_1 \underline{a}\,\underline{t}\, ds = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) = \varepsilon_{1jk} \frac{\partial a_k}{\partial x_j}$$

$$= \varepsilon_{1jk}\, a_{k,j}$$

We can show that for a triangle



$$\oint \underline{a}\,\underline{t}\, ds = \varepsilon_{1jk} \frac{\partial a_k}{\partial x_j} \left(\tfrac{1}{2} ds^2 \cos\theta \sin\theta\right) + O(ds^3)$$

do it. It might be included in the exam.