Taylor and Laurent Series

Series $\quad S = \sum_{n=1}^{\infty} c_n$ ← complex numbers

Convergence Theorem

S converges $\Longleftrightarrow$ S_n is a Cauchy sequence

$($for $\varepsilon > 0$ $\exists\, N(\varepsilon) \ni |S_m - S_n| < \varepsilon \;\; \forall m, n > N$

Necessary (but not sufficient) condition

$c_n \to$ as $n \to \infty$

Proof: set $m = n-1 \implies |S_m - S_n| = |c_n|$

Comparison Test

Let $M_n \geq 0$. If $\sum_{n=1}^{\infty} M_n$ converges and $|c_n| < M_n \implies$

$\sum c_n$ converges!

↑ This means $M_n \in \mathbb{R}$

<u>Proof</u>: let $\varepsilon > 0 \Rightarrow N_0$ is such that

$$M_p + M_{p+1} - - - + M_{p+q} < \varepsilon \quad \forall p > N_0 \quad \forall q > 0$$

$$\Rightarrow |c_p + c_{p+1} + - - c_{p+q}| \leq |c_p| + |c_{p+1}| + - |c_{p+q}|$$

$$\leq M_p + - - M_{p+q} < \varepsilon \quad \forall p > N_0$$

QED

<u>Example</u>: $\sum \frac{i^n}{n!} = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + - -$

Since $\left|\frac{i^n}{n!}\right| = \frac{1}{n!} < \frac{1}{2^n} \quad \forall n \geq 4$

$\Rightarrow$ Since $\sum \frac{1}{2^n}$ is a convergent geometric series then $\cdot \sum \frac{i^n}{n!}$ <u>converges</u>.

<u>Ratio Test</u> If $\lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| = L \Rightarrow \sum c_n$ converges if $L < 1$ and diverges if $L > 1$

Proof: Let $L<1$. Let $L<p<1$

$\Rightarrow \left|\frac{c_{n+1}}{c_n}\right| < p$ for some $n \geq N$

$$\Rightarrow |c_{N+1}| < p|c_N|$$
$$|c_{N+2}| < p|c_{N+1}| < p^2|c_N|$$
$$|c_{N+3}| < p|c_{N+2}| < p^3|c_N|$$

and so on..

$\sum c_n$ converges because by the comparison test $|c_n| < p^n|c_N|$ (we are comparing $\sum c_n$ with $\sum |c_N| p^n$

QED

POWER SERIES

$$S = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = L$$

Ratio Test: S converges ~~...~~ $\forall z \in |z-a| < \frac{1}{L}$ and diverges $\forall z \in |z-a| > \frac{1}{L}$. If $L=\infty$ series converges for $z=a$, and if $L=0$ series converges $\forall z$

Example: Geometric series $\sum z^n$

$a_n = 1 \ \forall n \Rightarrow L = 1 \Rightarrow$ series converges $\forall z \ni |z| < 1$

Proof: Use ratio test

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{a_{n+1}}{a_n} \frac{(z-a)^{n+1}}{(z-a)^n}\right| = \left|\frac{a_{n+1}}{a_n}(z-a)\right|$$

$$\lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| = |z-a| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |z-a| L$$

QED

Taylor Series

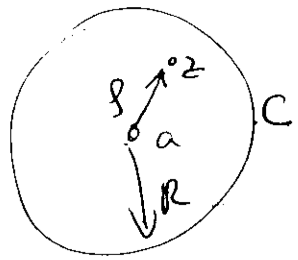
$f(z)$ analytic in D. Let $a \in D$ and $|z-a| < R$

$\Rightarrow f(z)$ admits precisely one power series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

PROOF:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z}\, d\xi$$



But $\frac{1}{\xi - z} = \frac{1}{\xi - a} \cdot \frac{1}{1 - \frac{z-a}{\xi - a}} =$

$$= \frac{1}{\xi - a} \cdot \frac{1}{1-t}$$

But $1 = (1 + t + t^2 + \dots t^n) - (t + t^2 + \dots t^n)$

$$1 = 1 + t + t^2 + \dots t^n - t(1 + t + \dots t^{n-1})$$

$$= (1 + t + t^2 + \dots t^{n-1})(1-t) + t^n$$

$$\Rightarrow \frac{1}{(1-t)} = 1 + t + t^2 + t^3 + \dots t^{n-1} + \frac{t^n}{1-t}$$

$$\Rightarrow \oint_C \frac{f(\xi)}{\xi - z}\, d\xi = \oint_C \frac{f(\xi)}{(\xi - a)} \left(1 + t + t^2 + \dots t^{n-1} + \frac{t^n}{(1-t)}\right) d\xi$$

$$= \oint_C \frac{f(\xi)}{(\xi - a)} \left\{ 1 + \frac{z-a}{\xi - a} + \frac{(z-a)^2}{(\xi - a)^2} + \dots + \frac{(z-a)^{n-1}}{(\xi - a)^{n-1}} + \frac{(z-a)^n}{(\xi - a)^n (1-t)} \right\} d\xi$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\left\{\oint_C \frac{f(\xi)}{(\xi-a)}\,d\xi + \oint_C \frac{f(\xi)}{(\xi-a)^2}(z-a)\,d\xi + \cdots + \oint_C \frac{f(\xi)}{(\xi-a)^n}(z-a)^{n-1}\,d\xi\right\}$$

$$+ \underbrace{\frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi-a)^{n+1}}\frac{(z-a)^n}{(1-t)}\,d\xi}_{R_n(z)}$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i}\sum_{k=0}^{n}(z-a)^k \underbrace{\oint_C \frac{f(\xi)}{(\xi-a)^{k+1}}\,d\xi}_{= \frac{2\pi i}{n!} f^{(k)}(a)} + R_n(z)}$$

$\Leftarrow$ Cauchy integral formula

$$\Rightarrow f(z) = \sum_{k=0}^{n} \frac{(z-a)^k}{n!} f^{(k)}(a) + R_n(z)$$

As $n \to \infty$ what happens to $R_n(z)$?

$$|R_n(z)| = \frac{\rho^n}{2\pi}\left|\oint_C \frac{f(\xi)}{(\xi-a)^n}\left\{\frac{1}{(\xi-z)}\right\}d\xi\right| \qquad = \left\{\frac{(\xi-a)}{1-t}\right\}$$

$$\le \frac{\rho^n}{2\pi}\left(\frac{M}{R^n(R-\rho)}\right)\underbrace{2\pi R}_{L} = \frac{MR}{R-\rho}\left(\frac{\rho}{R}\right)^n \to 0 \text{ as } n\to\infty$$

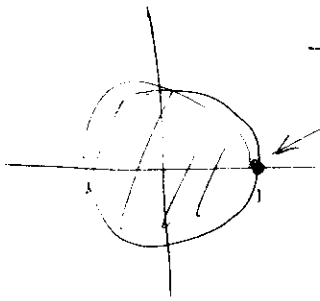
↑ML Bound

M is max $f(z)$ in circle

QED

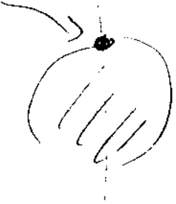
Radius of Convergence?

According to the theorem, in every disk of radius $R \ni f(z)$ is analytic



$\frac{1}{1-z}$

singular



Singular here. $\frac{1}{1+z^2}$

Product of Series

$$\left(\sum_{n=0}^{\infty} a_n (z-a)^n\right) \left(\sum_{n=0}^{\infty} b_n (z-a)^n\right) =$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)(z-a) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(z-a)^2 + \dots$$

$$= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)(z-a)^n$$

converges to $f(z)g(z)$ in $|z-a| < \mathrm{Min}(R_1 R_2)$

radius of conv. of f & g.

Linear Combination

$$\alpha \sum a_n (z-a)^n + \beta \sum b_n (z-a)^n =$$

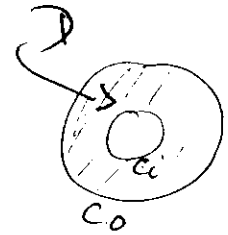
$$= \sum (\alpha a_n + \beta b_n)(z-a)^n$$

converges in $|z-a| < \mathrm{Min}(R_1 R_2$

Laurent Series

$f(z)$ analytic in D $\Rightarrow$

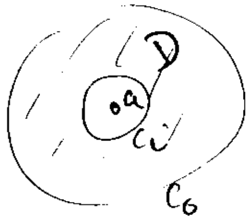
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \text{in}$$



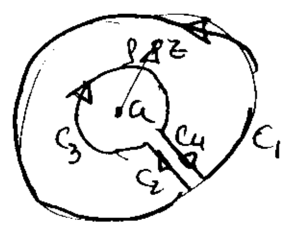
$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi-a)^{n+1}}\, d\xi$$

counter clockwise closed & simple

PARTIAL PROOF:



$\Rightarrow$



$$f(z) = \frac{1}{2\pi i} \oint_{C_1+C_2+C_3+C_4} \frac{f(\xi)}{\xi - z}\, d\xi =$$

$$= \frac{1}{2\pi i} \left\{ \oint_{C_1} + \oint_{C_3} \right\}$$

but $\dfrac{1}{\xi - z} = \dfrac{1}{\xi - a} \cdot \dfrac{1}{1-t} = \dfrac{1}{\xi - a} \displaystyle\sum_{n=0}^{\infty} \underbrace{\left(\frac{z-a}{\xi - a}\right)}_{t}^{n}$

$t = \left(\dfrac{z-a}{\xi - a}\right)$

converges if $\left|\dfrac{z-a}{\xi-a}\right| < 1$

$$\Rightarrow \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi = \oint_{C_1} \frac{f(\xi)}{\xi - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\xi - a}\right)^n d\xi$$

$$= \sum_{n=0}^{\infty} (z-a)^n \oint_{C_1} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi$$

$$= 2\pi i \sum_{n=0}^{\infty} c_n (z-a)^n.$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=0}^{\infty} c_n (z-a)^n$$

But C_1 can be deformed. $\Rightarrow$

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi$$

any C in D

Now consider $\oint_{C_3} \cdots$

Here we use. $\frac{1}{\xi - z} = \frac{-1}{(z-a)} \cdot \frac{1}{1 - \frac{\xi - a}{z-a}} =$

$$= \frac{-1}{(z-a)} \sum_{n=0}^{\infty} \left(\frac{\xi-a}{z-a}\right)^n$$

converges because $\left|\frac{\xi - a}{z-a}\right| < 1$

$\Rightarrow$

$$\frac{1}{2\pi i}\oint_{C_3} \frac{f(\xi)}{\xi - z}\, d\xi = -\frac{1}{2\pi i}\oint_{C_3} \frac{f(\xi)}{z-a}\sum_{n=0}^{\infty}\left(\frac{\xi-a}{z-a}\right)^n d\xi$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty} \left\{\oint_{C_3} f(\xi)(\xi-a)^n\, d\xi\right\}(z-a)^{-(n+1)}$$

$$\boxed{k=-(n+1)}$$

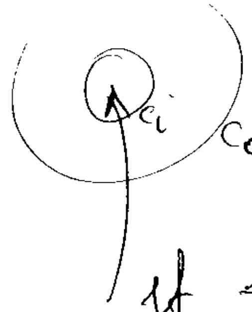
$$= \sum_{k=-1}^{-\infty}\left\{\frac{1}{2\pi i}\oint_{C_3}\frac{f(\xi)}{(\xi-a)^{k+1}}\, d\xi\right\}(z-a)^k$$

$$= \sum_{m=-1}^{-\infty}\underbrace{\left\{\frac{1}{2\pi i}\oint_{C} \frac{f(\xi)}{(\xi-a)^{n+1}}\, d\xi\right]}_{c_n}(z-a)^k$$

We rewrite the Laurent series as follows.

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=-1}^{-\infty} c_n (z-a)^n$$

$$= \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}$$



if $f(z)$ is analytic inside C_i

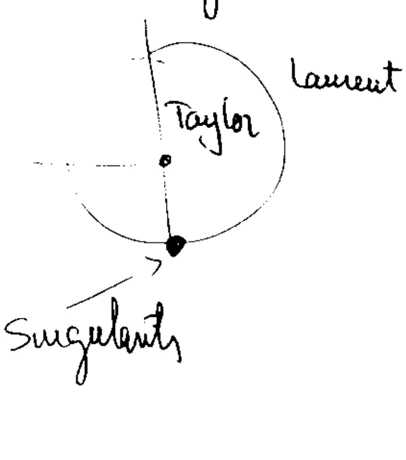
$$\Rightarrow \quad c_n = \begin{cases} \dfrac{f^{(n)}(a)}{n!} & n = 0, 1, 2 \\ 0 & n < 0 \end{cases}$$

Reducing everything to a Taylor series!

Example: All expansions of

$$f(z) = \frac{1}{z+i} \quad \text{about } z = 0$$

Singularities? $z = -i$



Taylor series in $|z|<$

$$\frac{1}{z+i} = \frac{1}{i} \cdot \frac{1}{1 + z/i} = -i \frac{1}{1 - iz}$$

$$= -i \left[1 + iz + (iz)^2 + \cdots \right]$$

$$= -i + z + iz^2 - \cdots$$

in $|z| < 1$

Laurent in $1 < |z| < \infty$

$$\frac{1}{z+i} = \frac{1}{z} \cdot \frac{1}{1+i/z} = \frac{1}{z} \frac{1}{1+t} = \frac{1}{z}\left(1-t+t^2+\cdots\right.$$

$$= \frac{1}{z}\left(1 - \frac{i}{z} - \frac{1}{z^2} + \cdots\right)$$

See more examples in text.