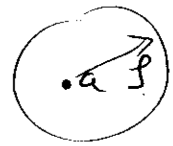
SINGULARITIES AND RESIDUE THEOREM

Let $f(z)$ singular at $z=a$



If $f(z)$ is analytic in a circle of radius ρ around a, (for some ρ) $\Rightarrow$

<u>a is an isolated singularity</u>

(which are the ones we will deal with)

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n =$$

$$= \dots + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \dots$$

Then if

$$f(z) = c_{-N}\frac{1}{(z-a)^N} + c_{-N+1}\frac{1}{(z-a)^{N-1}} + \dots$$

a is N^{th} order <u>pole</u>

if no finite N exists then the singularity is <u>essential</u>

Example $a = 0$

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2.$$

in $0 < |z| < 1$

$$\frac{1}{z^2(1-z)} = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots$$

in $1 < |z| < 1.$

$\Rightarrow$ Second order pole at $z = 0$

$\Rightarrow$ For the classification use expansion valid around the point.

Behaviour near poles

Suppose N^{th} order pole

$$f(z) = \frac{c_{-N}}{(z-a)^N} + \cdots + \frac{c_{-1}}{(z-a)} + \underbrace{c_0 + c_1(z-a) + \cdots}_{\substack{g(z) \\ \text{(analytic)}}}$$

$$\Rightarrow (z-a)^N f(z) = c_{-N} + c_{-N+1}(z-a) + \cdots + c_{-1}(z-a)^{N-1} + (z-a)^N g(z)$$

$$\rightarrow c_{-N} \text{ as } z \rightarrow a.$$

$\Rightarrow f(z) \sim \frac{c_{-N}}{(z-a)^N}$ near a.

We omit studying essential singularities.

RESIDUE THEOREM

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} c_{-1}^{(j)}$$

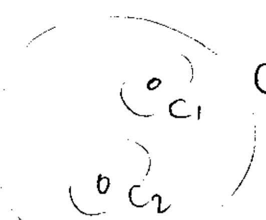
residues

$$c_{-1}^{(i)} = \frac{1}{2\pi i}\oint_{c_i} f(\xi)\,d\xi$$

around singular point

PROOF:

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$



Expand $f(z)$ in Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(1)} (z - z_1)^n$$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(2)} (z - z_2)^n$$

around z_1 and z_2

$$\Rightarrow \oint_{C_i} f(z)\,dz = \sum_{n=-\infty}^{\infty} c_n^{(i)} \underbrace{\oint (z-z_i)^n\,dz}_{=0 \text{ for } n\neq -1}$$

$$= 2\pi i\, c_{-1}^{(i)}$$

QED

How to calculate residues?

Let $f(z)$ have a first order pole

$$\Rightarrow f(z) = c_{-1}\frac{1}{z-a} + c_0 + c_1(z-a) + \cdots$$

in $0 < |z-a| < \rho$ for some ρ

$$\Rightarrow (z-a)f(z) = c_{-1} + c_0(z-a) + \cdots$$

$$\Rightarrow c_{-1} = \lim_{z\to a}\left[f(z)(z-a)\right]$$

Nth order pole?

$$c_{-N} = \lim_{z\to a}\left[f(z)(z-a)^N\right]$$

$$\boxed{C_{-1} = \frac{1}{(N-1)!} \lim_{z \to a} \left[\frac{d^{N-1}}{dz^{N-1}} \left[(z-a)^N f(z) \right] \right]}$$

indeed

$$f(z)(z-a)^N = C_{-N} + (z-a) C_{-N+1} + \cdots (z-a)^{N-1} C_{-1} + (z-a)^N C_0 + \cdots$$

$$\frac{d}{dz}\left[f(z)(z-a)^N \right] = C_{-N+1} + \cdots + (N-1)(z-a)^{N-2} C_{-1} + N(z-a)^{N-1} C_0$$

and so on

Example:

$$f(z) = \frac{1}{(z+4)(z-1)^3}$$

first order pole at $z=-4$
third order pole at $z=1$

$$\operatorname*{Res}_{z=-4} f = \lim_{z \to -4} \left[\frac{\cancel{(z+4)} \cdot 1}{\cancel{(z+4)} (z-1)^3} \right] = -\frac{1}{125}$$

$$\operatorname*{Res}_{z=1} f = \frac{1}{2!} \lim_{z \to 1} \left\{ \frac{d^2}{dz^2} \cancel{(z-1)^3} \frac{1}{(z+4)\cancel{(z-1)^3}} \right\} =$$

$$= \frac{1}{125}$$

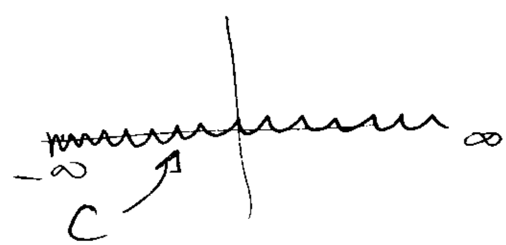
EVALUATION OF REAL INTEGRALS

$$I = \int_{-\infty}^{\infty} f(x)\, dx$$

Let $f(x) = \dfrac{1}{x^2+1}$

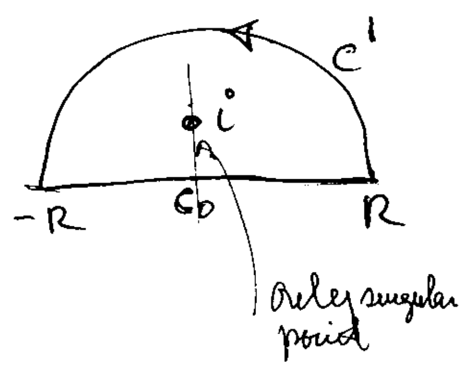
To do this write

$$I = \int_C \frac{dz}{z^2+1}$$



Consider

$$J = \int_{C' + C_0} \frac{dz}{z^2+1}$$



$$J = 2\pi i \operatorname*{Res}_{z=i} f = 2\pi i \lim_{z \to i} \left[\cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}} \right]$$

$$= \pi$$

on the other hand

$$J = \pi = \oint_{C_0} + \oint_{C'} = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C'} \frac{dz}{z^2+1}$$

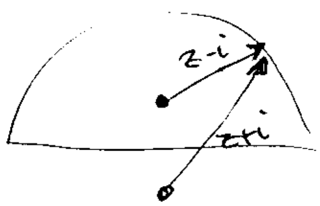
$$\Rightarrow \int_{-R}^{R} \frac{dx}{x^2+1} = \pi - \int_{C'} \frac{dz}{z^2+1}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^2+1} =$$

$$= \pi - \lim_{R \to \infty} \int_{C'} \frac{dz}{z^2+1}$$

We can use the ML Bound

$$\left|\frac{1}{z^2+1}\right| = \frac{1}{|z-i|\,|z+i|} \leq \frac{1}{(R-1)\sqrt{R^2+1}}$$



$|z-i|$ is smallest when $z=Ri$
$|z+i|$ is smallest when $z=\pm R$

$$\Rightarrow \left|\frac{1}{z^2+1}\right| \leq \frac{1}{(R-1)\sqrt{R^2+1}}$$

$$\Rightarrow \left|\int_{C_R}\frac{1}{z^2+1}\,dz\right| \leq \frac{1}{(R-1)\sqrt{R+1}}\overbrace{\pi R}^{\text{Length}} \sim \frac{\pi}{R}$$

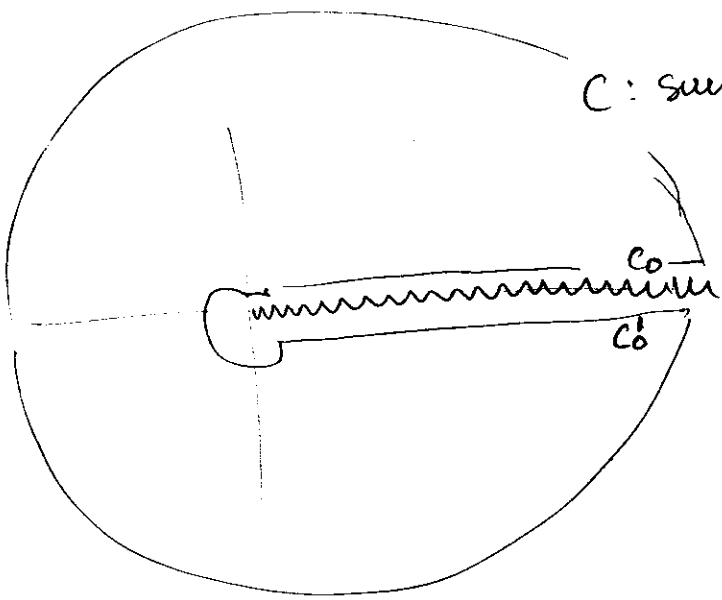
$\Rightarrow$ As $R\to\infty$ the integral vanishes

Another problem

$$I = \int_0^\infty \frac{x^{1/3}}{(x+1)^2}\, dx$$

$\Rightarrow$ Consider $J = \oint_C \frac{z^{1/3}}{(z+1)^2}\, dz$

C: suitable chosen

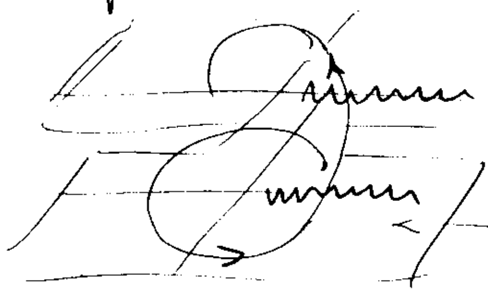


$z^{1/3}$ is multiple valued.

That is $8^{1/3} = 2, 2e^{i2\pi/3}, 2e^{i4\pi/3}, 2e^{i6\pi/3}$, etc

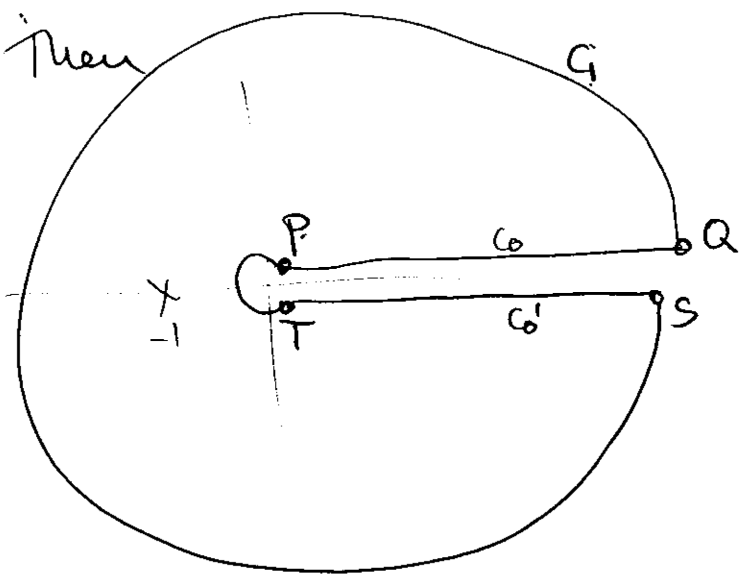
but only 3 values represent $\neq$ points in the complex plane. That is why we need to go around C_0 an C_0' because on the upper part $8^{1/3} = 2$, but on the lower part of C_0 (on C_0') $8^{1/3} = 2 \cdot e^{i6\pi/3}$. We call the line dividing multiple values from each other. "Branch cuts"

It is like having multiple image complex planes in a "pancake"



after 360° one should "jump" to the next branch.

Then



$$J = \underbrace{\int_{C_0}}_{} = 2\pi i \sum \underbrace{c_{-1}^{i}}_{}$$

We are interested in this value

singularity is at $z=-1$ (second order pole)

On C_0 $(\overline{PQ})$ $\quad z = xe^{i0}$

$$\Rightarrow z^{1/3} = x^{1/3} e^{i0/3} = x^{1/3}$$

on C_0' (ST) $\quad z = xe^{2\pi i}$

$$\Rightarrow z^{1/3} = x^{1/3} \cdot e^{2\pi i/3}$$

$$\Rightarrow 2\pi i \operatorname*{Res}_{z=-1} f(z) = \int_\varepsilon^R \frac{x^{1/3}}{(x+1)^2} dx + \int_{C_1} \frac{z^{1/3}}{(z+1)^2} dz$$

$$+ \int_R^\varepsilon \frac{x^{1/3} e^{2\pi i/3}}{(x+1)^2} dx + \int_{C_\varepsilon} \frac{z^{1/3}}{(z+1)^2} dz$$

$\int_{C_1} \to 0$ as $R \to \infty$ (Apply ML bound

$$\left| \frac{z^{1/3}}{(z+1)^2} \right| = \frac{R^{1/3}}{|z+1|^2} = \frac{R^{1/3}}{|z-(-1)|^2} \le \frac{R^{1/3}}{(R-1)^2}$$

$$\left| \int_{C_1} \frac{z^{1/3}}{(z+1)^2} dz \right| \le \frac{R^{1/3}}{(R-1)^2} \underbrace{2\pi R}_{L} \sim \frac{2\pi}{R^{2/3}} \to 0 \text{ as } R \to \infty$$

$$\left| \int_{C_\varepsilon} f(z)\, dz \right| \leq \frac{\varepsilon^{1/3}}{(1-\varepsilon)^2} 2\pi\varepsilon \sim 2\pi \varepsilon^{4/3} \to 0$$

as $\varepsilon \to 0$

$$\Rightarrow \int_0^\infty \frac{x^{1/3}}{(x+1)^2} dx + \int_\infty^0 \frac{x^{1/3} e^{2\pi i/3}}{(x+1)^2} dx =$$

$$= 2\pi i \operatorname*{Res}_{z=-1} f(z)$$

Since the pole is order 2 -

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz}\left[\cancel{(z+1)^2} \frac{z^{1/3}}{\cancel{(z+1)^2}} \right]$$

$$= \frac{1}{3}(-1)^{-2/3} = \frac{e^{-2\pi i/3}}{3}$$

$$\Rightarrow \left(1 - e^{2\pi i/3}\right) \cdot \int_0^\infty \frac{x^{1/3}}{(x+1)^2} dx = \frac{e^{-2\pi i/3}}{3}$$

or $\boxed{\int_0^\infty \frac{x^{1/3}}{(x+1)^2} dx = \frac{2\pi}{3\sqrt{3}}}$

What to do when $\sin\theta$ $\cos\theta$ show up.

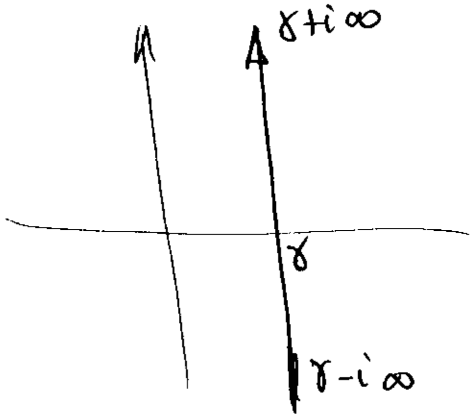
write $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2} = \dfrac{z - z^{-1}}{2}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

Inverse Laplace Transform

We saw that

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}\,ds$$



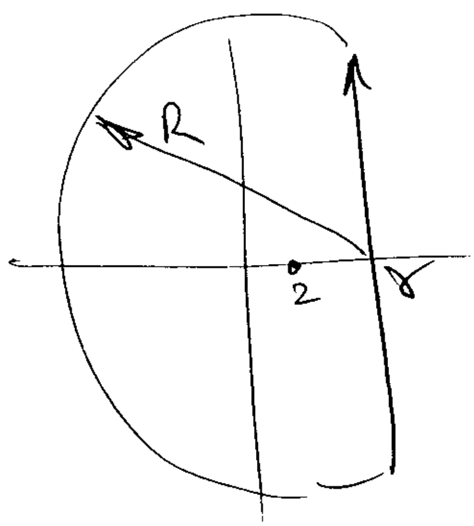
we need $\operatorname{Re} s = \gamma > c$

c satisfying

$|f(t)| \le K e^{ct}$ $\forall t > T$

some T

Let $F(s) = \dfrac{1}{(s-2)^3}$



$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{(s-2)^3}\,ds$$

What γ should we use.

$\gamma > c$? But what is

c.

Here we state that we pick γ to the right of all singularities of F(s). Otherwise, could do a trial and error?

In this case $\gamma > 2$

To evaluate f(t) we use

$$I = \frac{1}{2\pi i}\oint \frac{e^{zt}}{(z-2)^3}\,dz = f(t) + \int_{C_R} \frac{e^{zt}}{(z-2)^3}\,dz$$

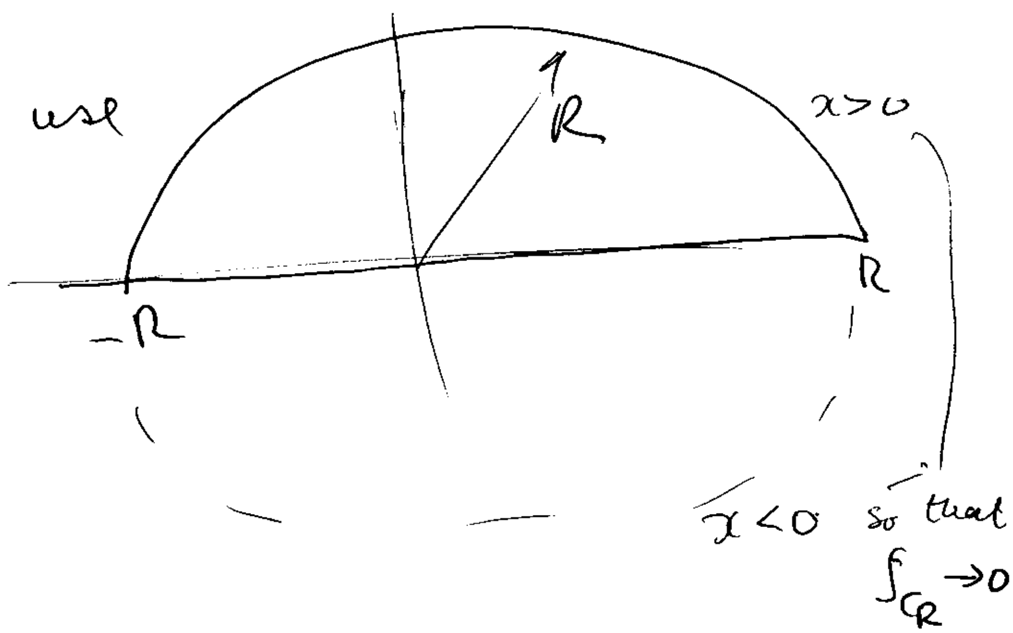
$$= 2\pi i \operatorname*{Res}_{z=2}\left\{\frac{e^{zt}}{(z-2)^3}\right\}$$

Similarly the Fourier Transform

$\hat{f}(\omega)$. has an inverse

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, dx$$

We use



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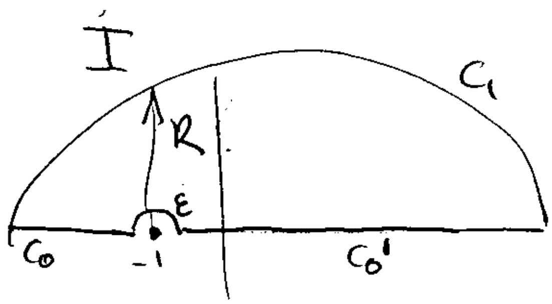
Singularities in integrand

$$\int_{-\infty}^{\infty} \frac{1}{x+1}\,dx = I$$

let



$$J = \int_{C_0} + \int_{C_\epsilon} + \int_{C_0'} + \int_{C_1}$$

As usual when $R \to \infty$ $\int_{C_1} \to 0$

$$\Rightarrow \boxed{\underbrace{\int_{C_0} + \int_{C_0'}}_{I} + \int_{C_\epsilon} = \underbrace{2\pi i \sum_j \mathrm{Res}_j}_{\text{We know how to do this!}}}$$

$$\int_{C_\epsilon} = \int_{-\pi}^{0} \frac{1}{(\epsilon e^{i\theta}+1)}\,d\theta = \left[\theta - \frac{\ln(1+\epsilon e^{i\theta})}{i}\right]_{-\pi}^{0}$$

$$= \pi + i\left[\ln\frac{(1+\epsilon)}{(1-\epsilon)}\right] \xrightarrow[\epsilon \to 0]{} \pi$$