Power Series Solutions

Consider

$$y'' + p(x)y' + q(x)y = 0$$

If p and q are analytic at x_0

$\Rightarrow$ a solution of the form

$$y(x) = \sum_0^\infty a_n (x-x_0)^n$$

exists.

f is analytic if its Taylor series exists in an interval and converges to f.

Methodology - propose $y(x) = \sum_0^\infty a_n (x-x_0)^n$ and replace in equation!

Example $\quad y'' + y = 0 \qquad x_0 = 0$

$$y' = \sum_1^\infty n a_n x^{n-1} \qquad y'' = \sum_2^\infty n(n-1) a_n x^{n-2}$$

$$y'' + xy = \sum_2^\infty n(n-1) a_n x^{n-2} + \sum_0^\infty a_n x^n = 0$$

or $\displaystyle\sum_{m=0}^\infty \{(m+2)(m+1)a_{m+2} + a_m\} x^m = 0$

$$\boxed{(m+2)(m+1)\, a_{m+2} + a_m = 0}$$

$$\boxed{a_{n+2} = -\frac{a_n}{(n+2)(n+1)}}$$

$a_2 = -\frac{1}{2} a_0$

$a_3 = -\frac{1}{6} a_1$

$a_4 = -\frac{1}{12} a_2 = \frac{1}{4!} a_0$

etc

$\Rightarrow$ a_0, a_1 undefined

Use IC to determine a_0, a_1

$$y(x) = a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4$$

$$= a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots\right) + a_1 \left(x - \frac{1}{3!} x^3 + \cdots\right)$$

$$\boxed{y(x) = a_0 y_1(x) + a_1 y_2(x)}$$

Watch change of exponents in sum

$$y'' + xy = 0$$

$$y'' + xy = \sum_2^\infty n(n-1) a_n x^{n-2} + \sum_0^\infty a_n x^{n+1} = 0$$

$$= \sum_0 (m+2)(m+1) a_{m+2} x^m + \underbrace{\sum_1^\infty a_{m-1} x^m}_{\sum_0^\infty a_{n-1} x^n} = 0 \quad / \boxed{a_{-1} \equiv 0}$$

Singular points
Method of Frobenius

Regular singular point $\left.\begin{array}{l}(x-x_0)\,p(x)\\(x-x_0)^2 q(x)\end{array}\right\}$ analytica

Irregular " " $\rightarrow$ not regular!

We deal with regular singular points

$$y'' + p(x)y' + q(x)y = 0$$

$x=0$ is a regular singular point

$$\Downarrow$$

$$x^2y'' + x\,[\underbrace{x\,p(x)}]\,y' + [x^2q(x)]\,y = 0$$

analytical

If not change variable

$\Rightarrow$

$$x^2y'' + x(p_0 + p_1x + \cdots)y' + (q_0 + q_1x + \cdots)y =$$

Close to zero

$$x^2y'' + p_0xy' + q_0y = 0$$

Cauchy Euler

solution is of the form x^r

$\Rightarrow$ Propose

$$y(x) = x^r\,(a_0 + a_1 x + a_2 x^2 \cdots)$$

Now need to determine $r,\ a_0, a_1 \cdots$

Proceed as usual!

Example. $6x^2y'' + 7xy' - (1+x^2)y = 0$

$x=0$ is regular singular (why?)

$$6x^2 \sum_0^\infty (n+r)(n+r-1)\,a_n x^{n+r-2} + 7x \sum_0^\infty (n+r)\,a_n x^{n+r-1}$$
$$-(1+x^2)\sum_0^\infty a_n x^{n+r} = 0$$

$$\sum_0 \left\{6(n+r)(n+r-1) + 7(n+r) - 1\right]a_n x^{n+r} - \sum_{n=2}^\infty a_{n-2}x^{n+r} = 0$$

Set $a_{n-2} \equiv a_{n-1} \equiv 0$

$$\Rightarrow \sum_0 \left\{\left[6(n+r)(n+r-1) + 7(n+r) - 1\right]a_n - a_{n-2}\right\}x^{n+r} = 0$$

$$\boxed{a_n = \frac{a_{n-2}}{6(n+r)(n+r-1) + 7(n+r) - 1}}$$

$a_0 = 0$? no. $a_0 \neq 0$

$$\left[6(n+r)(n+r-1) + 7(n+r) - 1\right]_{n=0} = 0$$
$$\Downarrow$$
$$6r^2 + r - 1 = 0$$

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$$r = \begin{cases} -1/2 \\ 1/3 \end{cases}$$

$$\Rightarrow \quad a_n = \frac{1}{6(n-1/2)^2 + n - \frac{3}{2}} \; a_{n-2}$$

$a_1 = a_{-1} = 0$

$a_2 = \frac{1}{14} a_0$

$\vdots$

Set $r = 1/3$

$$a_n = \frac{1}{6(n+1/3)^2 + n - \frac{2}{3}} \; a_{n-2}$$

$$y_1 = a_0 x^{-1/2} \left[1 + \frac{1}{14} x^2 + \frac{1}{(76)(14)} x^4 + \cdots \right]$$

$$y_2 = a_0 x^{1/3} \left[1 + \frac{1}{34} x^2 + \frac{1}{(116)(34)} x^4 + \cdots \right]$$

$\Downarrow$

Why pursue this around $x=0$

Why not use a regular point ($x=1$ or other)

Answer: would not capture right behaviour around zero.

Generalizing

$$x^2 y'' + x(p_0 + p_1 x + \cdots) y' + (q_0 + q_1 x + \cdots) y = 0$$

$$\Downarrow$$

$$\sum_0^\infty (n+r)(n+r-1) a_n x^{n+r} + (p_0 + p_1 x \cdots) \sum_0^\infty (n+r) a_n x^{n+r}$$
$$+ (q_0 + q_1 x \cdots) \sum a_n x^{n+r} = 0$$

Assuming $a_0 \neq 0 \Rightarrow \boxed{r^2 + (p_0 - 1) r + q_0 = 0}$

Indicial equation

Once 2 values of r are known one can calculate all coefficients as a function of a_0.

what if $r_1 = r_2$?

use reduction of order.

$$y_1 = x^r \sum a_n x^n$$
$$y_2 = \ln x \, y_1 + x^r \sum_1^\infty c_n x^n$$

Proof:

$y_2 = A(x)\, y_1(x)$

$$y_2'' + p(x)\, y_2' + q(x)\, y_2 = A'' y_1 + A'(2y_1' + py_1) + A\underbrace{(y_1'' + py_1' + qy_1)}_{=0} = 0$$

$$\Rightarrow A'' y_1 + A'(2y_1' + py_1) = 0$$

$$\frac{dA'}{A'} + 2\frac{dy_1}{y_1} + p\,dx = 0$$

$$\ln A' + 2\ln y_1 + \int p\,dx = \ln C$$

$$\Rightarrow \ln \frac{A' y_1^2}{C} = -\int p(x)\,dx$$

$$A' = \frac{C\, e^{-\int p(x)dx}}{y_1^2} = \frac{C\, e^{-\int p(x)dx}}{[x^r(1+a_1x+\cdots)]^2}$$

$$= C\,\frac{e^{-\int \frac{p_0}{x}dx + \int (p_1 + p_2 x \cdots)dx}}{x^{2r}[1+a_1x+\cdots]^2}$$

$(-p_0 \ln x)$

$$e^{-\int \frac{p_0}{x} dx} = x^{-p_0}$$

$e^{-\int (p_1 + p_2 x + \dots) dx}$ is analytical

$$\Rightarrow \frac{e^{-\int p_1 + p_2 x + \dots}}{[1 + a_1 x + \dots]^2} = \frac{e^{-\int p_1 + p_2 x - \dots}}{[1 + 2a_1 x + \dots]}$$

$$= 1 + K_1 x + K_2 x^2 + \dots$$

$$\Rightarrow A'(x) = C \frac{1}{x^{2r+p_0}} (1 + K_1 x + K_2 x^2 + \dots)$$

but $2r + p_0 = 1$.

$$\Rightarrow A(x) = C(\ln x + K_1 x + \dots)$$

$$\Rightarrow y_2 = A(x)\, y_1(x) =$$

$$= (\ln x + K_1 x + \dots)\, y_1(x) =$$

$$= y_1 \ln x + (K_1 x + K_2 x^2 \dots) x^r (1 + a_1 x + \dots)$$

$$\boxed{y_2 = y_1 \ln x + x^r \sum_1^\infty c_n x^n}$$

What if $r_1 = r_2 + 1$?

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Consider:

$$\sum_0^\infty (n+r)(n+r-1)a_n x^{n+r} + (p_0 + p_1 x + \ldots)\sum (n+r)a_n x^{n+r}$$

$$+ (q_0 + q_1 x)\sum a_n x^{n+r} = 0$$

$$x^r : [r(r-1) + p_0 r + q_0]a_0 = 0$$

$$x^{r+1} : [(r+1)r + p_0(r+1) + q_0]a_1 + (p_1 r + q_1)a_0 = 0$$

$\vdots$

or

$$x^r : F(r) = 0$$

$$x^{r+1} : F(r+1)a_1 + (p_1 r + q_1)a_0 = 0$$

$\vdots$

Since

$r_1 = r_2 + 1$

$$\Rightarrow \quad F(r_2 + 1) = F(r_1) = 0$$

$$\Rightarrow 0a_1 + \underbrace{(p_1 r_2 + q_1)}_{\text{What if this is not zero?}} a_0 = 0$$

Can't find a solution

If $p_1 r_2 + q_1 = 0 \Rightarrow 0a_1 = 0$
a_1 is arbitrary

Then the solution is

$$y_1 = x^{r_1} \sum a_n x^n$$
$$y_2 = K y_1(x) \ln x + x^{r_2} \sum_0^\infty d_n x^n$$

where K could be zero

Let's work one example out!

$$x y'' + y = 0 \qquad \therefore \; x^2 y'' + x y = 0$$

Indicial equation $\quad r^2 + (p_0 - 1) r + q_0 = 0$
$$r^2 + (-1) r + 0 = 0$$

$$r(r-1) = 0 \quad \begin{cases} r = 0 \\ r = 1 \end{cases}$$

Let us use $r_1 = 1$, the largest root.

$$\sum_0^\infty a_n x^{n+r-1} (n+r)(n+r-1) + \sum_0^\infty a_n x^{n+r} = 0$$

$$\Rightarrow \left[(n+r+1)(n+r)\, a_{n+1} + a_n \right] = 0$$

$$\Rightarrow a_{n+1} = -\frac{1}{(n+2)(n+1)} a_n \qquad (r = 1)$$

$$\Rightarrow a_n = -\frac{(-1)^n}{(n+1)(n!)^2} a_0$$

$$y_2(x) = K\, y_1(x) \ln x + x^{1} \sum_0^\infty d_n x^n$$

$$K x^2 y_1'' \ln x + 2K x y_1' - K y_1 + \sum_0^\infty (n-1)n\, d_n x^n$$
$$+ K x y_1 \ln x + \sum_0^\infty d_n x^{n+1} = 0$$

$$K y_1 - 2K x y_1' = \sum_0^\infty [n(n-1) d_n + d_{n-1}] x^n$$

$$\Rightarrow n(n-1) d_n + d_{n-1} = -K \frac{(-1)^{n-1}(2n-1)}{n[(n-1)!]^2}$$

$n=1 \quad d_0 = -K$

$n=2 \quad d_2 = \frac{3}{4} K - \frac{1}{2} d_1$

$n=3 \quad d_3 = -\frac{7}{36} K + \frac{1}{12} d_1$

$$\sum_0^\infty d_n x^n = K\left(-1 + \frac{3}{4}x^2 - \frac{7}{36}x^4 + \cdots\right)$$
$$+ d_1 \underbrace{\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \cdots\right)}_{y_1(x)}$$

$$\Rightarrow d_1 = 0$$

$$y_2 = K\left[y_1(x) \ln x + \left(-1 + \frac{3}{4}x^2 - \frac{7}{36}x^4\right)\right.$$

Let's derive

$$y_2 = K\, y_1(x) \ln x + x^{r_2} \sum_0^\infty d_n x^n$$

$$y_2 = A\, y_1 \quad \Rightarrow \quad A' = c\, \frac{e^{-\int p(x)dx}}{y_1^2}$$

$$A' = c\, \frac{e^{-p_0 \ln x}\, e^{-(p_1 x + \cdots)}}{x^{2r_1} \cdot (1 + 2a_1 x + \cdots)}$$

$$= c\, \frac{1}{x^{2r_1 + p_0}} (1 + K_1 x + \cdots)$$

Indicial equation is

$$r^2 + (p_0 - 1) r + q_0 = 0$$

$$\Rightarrow r_{1,2} = \frac{1-p_0}{2} \pm \sqrt{\left(\frac{1-p_0}{2}\right)^2 - q_0}$$

$$r_1 = \frac{1-p_0}{2} + \sqrt{\left(\frac{1-p_0}{2}\right)^2 - q_0}$$

$$r_2 = \frac{1-p_0}{2} - \sqrt{\left(\frac{1-p_0}{2}\right)^2 - q_0}$$

But $r_1 - r_2 = m$

$$\Rightarrow 2\sqrt{\left(\frac{1-p_0}{2}\right)^2 - q_0} = m$$

$$\left(\frac{1-p_0}{2}\right)^2 - q_0 = \left(\frac{m}{2}\right)^2 \Rightarrow r_2 = \left(\frac{1-p_0}{2}\right) - \frac{m}{2}$$

$$r_1 = \left(\frac{1-p_0}{2}\right) + \frac{m}{2}$$

$$2r_1 = 1 - p_0 + m$$

$$2r_1 + p_0 = 1 + m$$

$$\Rightarrow A'(x) = \frac{C}{x^{1+m}} \left(1 + K_1 x + \cdots\right)$$

$$A(x) = C\left(\frac{-1}{m x^m} - K_1 \frac{1}{(m-1)x^{m-1}} \cdots K_m \ln x + K_{m+1} x + \cdots\right)$$

$$y(x) A(x) = C K_m y(x) \ln x + x^{r_1} \sum_0^\infty a_n x^n \left(\frac{-1}{m x^m} - K_1 \frac{1}{(m-1)x^{m-1}} - \frac{K_{m-1}}{x} + K_{m+1} x + \cdots\right)$$

$$= C K_m y(x) \ln x + x^{(r_1 - m)} \sum_0^\infty a_n x^n \left(-\frac{1}{m} - \frac{K_1}{m-1} x \cdots - K_{m-1} x^{m-1} + K_{m+1} x^{m+1} + \cdots\right)$$

r_2

QED

Legendre Functions

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$ ← Legendre equation

Singular points ± 1. However expansions around $x=0$ are.

$$y_1(x) = \left[1 - \frac{\lambda}{2}x^2 + \frac{(6-\lambda)\lambda}{24}x^4 - \cdots\right]$$

$$y_2(x) = \left[x + \frac{2-\lambda}{6}x^3 + \frac{(12-\lambda)(2-\lambda)}{120}x^5 + \cdots\right]$$

Radius of convergence $-1 < x < 1$.

When $\lambda = n(n+1)$ the series becomes a polynomial.

n	λ	$P_n(x)$
0	0	$P_0(x) \equiv 1$
1	2	$P_1(x) \equiv x$
2	6	$P_2(x) = \frac{1}{2}(3x^2-1)$

} Legendre polynomials

Properties. — $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[(x^2-1)^n\right]$

Rodrigues' formula

— $\int_{-1}^{1} P_j(x)\,P_k(x)\,dx = 0 \quad j \neq k$

orthogonal!

Bessell Functions

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$ Bessell equation of order ν

a) ν not integer.

$$y = \sum a_k x^{k+r} \quad a_0 \neq 0$$

$$\Rightarrow \sum \left\{ [(k+r)^2 - \nu^2] a_k + a_{k-2} \right\} x^{k+r} = 0$$

indicial equation is $r^2 - \nu^2 = 0$

$$\Rightarrow \boxed{r = \pm \nu}$$

$$a_k = -\frac{1}{k(k+2\nu)} a_{k-2}$$

After some manipulation one gets

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \underbrace{\Gamma(\nu+k+1)}_{\text{gama function}}} \left(\frac{x}{2}\right)^{2k}$$

$$\begin{cases} \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt & (x>0) \\ \Gamma(n+1) = n! \end{cases}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k}$$

b) ν integer $\quad \nu = n \quad (r = \pm n)$

$$y_1 = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

$y_2 = ?$

$$\boxed{y_2 = \underset{\substack{\uparrow \\ \text{recipe.}}}{k J_n(x) \ln x} + x^{-n} \cdot \sum_{0}^{\infty} d_n x^n}$$

Particular cases

$n=0$

$$\underbrace{y_2 = J_0(x) \ln x + \left(\frac{x}{2}\right)^2 - \left(1+\frac{1}{2}\right)\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \cdots}_{Y_0(x)}$$

↑ Neumann function of order zero

Weber proved that it is more convenient to use a linear combination of these two

$$y_2 = \frac{2}{\pi}\left[Y_0(x) + (\gamma - \ln 2) J_0(x)\right] = Y_0(x)$$

Weber's Bessel function of the second kind (order zero)

$\gamma = 0.5772157$ (Euler constant)

General Solutions

Define $Y_\nu(x) \equiv \dfrac{\cos\nu\pi \, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}$

$$\lim_{\nu \to n} Y_\nu(x) = Y_n(x)$$

Bessel function of the second kind of order n.

$$\Rightarrow \underbrace{y(x) = A\, J_\nu(x) + B\, Y_\nu(x)}_{\text{general solution}}$$

Other functions

$$\left\{ \begin{array}{l} H_\nu^{(1)}(x) \equiv J_\nu(x) + i\, Y_\nu(x) \\ H_\nu^{(2)}(x) \equiv J_\nu(x) - i\, Y_\nu(x) \end{array} \right.$$

Hankel Functions (useful in wave propagation)

$$y(x) = \underbrace{A\, H_\nu^{(1)}(x) + B\, H_\nu^{(2)}(x)}_{\text{general solution}}$$

Modified Bessel Equation

$$x^2y''+xy'+(-x^2-n^2)y=0$$

difference in sign!

Let $t=ix \to \left[\, y(x)=y(-it)=Y(t) \,\right]$

$$\Rightarrow t^2Y''+tY'+(t^2-n^2)Y=0$$

$$\Rightarrow y(x)=A\,J_n(ix)+B\,Y_n(ix)$$

But $J_n(ix)=\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{ix}{2}\right)^{2k+n}=$

$$= i^{n} \qquad \left(\frac{x}{2}\right)^{2k+n}$$

$$\Rightarrow \underbrace{I_n(x)}=i^{-n}J_n(ix)=\sum_{k=0}^{\infty}\frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}$$

Modified Bessel function of the first kind.

$$K_n(x)=\frac{\pi}{2}\,i^{n+1}\left[J_n(ix)+i\,Y_n(ix)\right]$$

Second kind

Differences