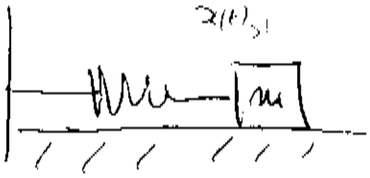
Phase Plane

MOTIVATION

Consider the spring system



$$m\ddot{x} + kx = 0$$

$$\Rightarrow x = A \sin(\omega t + \phi)$$

Now, express

$$\left.\begin{aligned} y &= \dot{x} \\ \ddot{x} = \frac{dy}{dx} &= -\frac{k}{m} x \end{aligned}\right| \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -\frac{k}{m} x \end{cases}$$

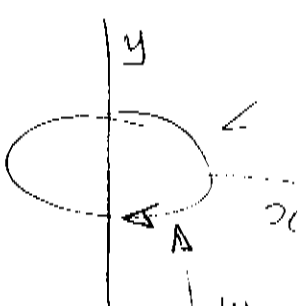
or divide $\dfrac{dy}{dx} = -\dfrac{k}{m}\dfrac{x}{y}$

$$\Rightarrow m y\, dy + k x\, dx = 0$$

$$\Rightarrow \tfrac{1}{2} m y^2 + \tfrac{1}{2} k x^2 = C$$

what does this mean in energy terms?

ellipse



We call this a trajectory in the phase plane!

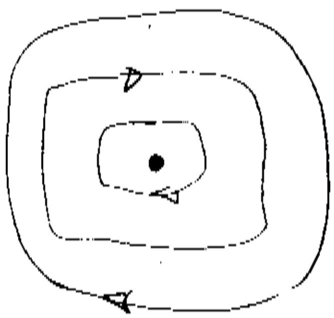
Assume now a hard spring
(grows stiffer as x increases)

$$x'' + x + x^3 = 0$$

$$\Rightarrow \quad \dot{x} = y$$
$$\dot{y} = -x - x^3$$

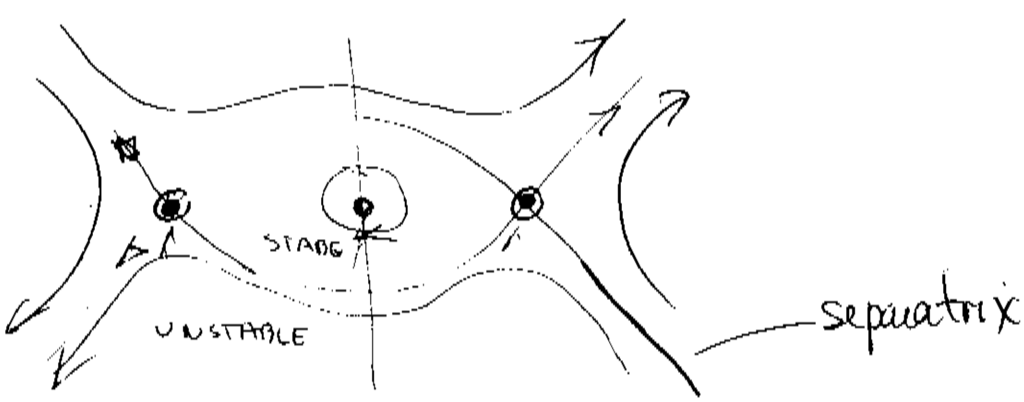
$$\frac{dy}{dx} = -\frac{x + x^3}{y} \Rightarrow y\,dy + (x + x^3)\,dx = 0$$

$$\Rightarrow \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = C$$



soft spring

$$x'' + x - x^3 = 0 \quad \Rightarrow \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 = C$$



Fixed points: places where $\boxed{\begin{aligned}\dot{x} &= 0\\ \dot{y} &= 0\end{aligned}}$

$(y, x) = (-1,0), (0,0), (0,1)$

Singular points

Consider $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$

(x_s, y_s) is a singular point if

$\begin{cases} f(x_s, y_s) = 0 \\ g(x_s, y_s) = 0 \end{cases}$ recall that $\dfrac{dy}{dt} = f(x,y)$, $\dfrac{dx}{dt} = g(x,y)$

$\Rightarrow$ singular points are fixed points!

Need to study the neighborhood.

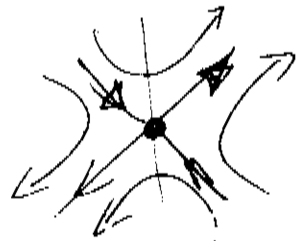
$\Rightarrow$ Make taylor expansion.

Consider $(x_s, y_s) = (1,0)$

$$\Rightarrow \begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 = 0 + 2(x-1) + \underbrace{\frac{6}{2!}(x-1)^2 +}_{\text{Taylor series}} \end{aligned}$$

$$\Rightarrow \begin{aligned} \dot{x} &= y \\ \dot{y} &= 2(x-1) \end{aligned} \quad \xrightarrow[\substack{Y = y \\ X = x-1}]{} \quad \boxed{\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= 2X \end{aligned}}$$

$$\Rightarrow \frac{dY}{dX} = \frac{2X}{Y} \Rightarrow Y^2 = 2X^2 + C$$



$\begin{pmatrix} x_s = 1 \\ y_s = 0 \end{pmatrix}$

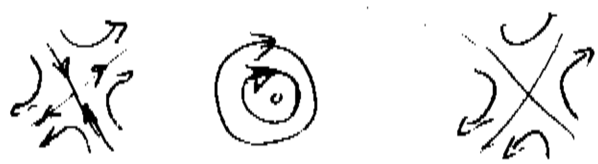
Same is for $x_s = -1$
$y_s = 0$

Near zero $\quad \dot{x} = y$
$\dot{y} = -x$

$\dfrac{dy}{dx} = -\dfrac{x}{y}$

$\Rightarrow y^2 + x^2 = C$

circles



Classification

After linearization

$$X' = aX + bY$$
$$Y' = cX + dY$$

$$\begin{cases} X = x - x_s \\ Y = y - y_s \end{cases}$$

FIXED POINTS

$$X' = 0 = aX + bY$$
$$Y' = 0 = cX + dY$$

$\Downarrow$

has unique solution

$\Longleftrightarrow ad - bc \neq 0$

Otherwise ∞ solutions (lines)

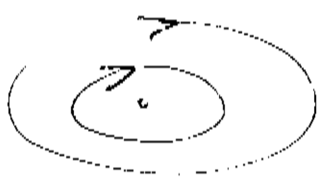
Solve $X' = aX + bY$
$Y' = cX + dY$

and get

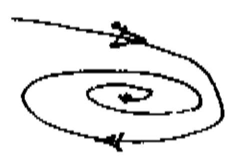
$$X = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$Y = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}$$

λ_1, λ_2 roots of $\lambda = \underbrace{\frac{a+d}{2}}_{\mathrm{tr}A} \pm \sqrt{\frac{(a-d)^2}{4} + bc}$

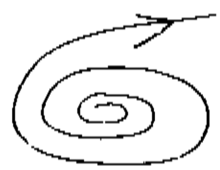
$\Rightarrow$ Cases
a) $\lambda_1 \lambda_2$ imaginary $\Rightarrow$ Center
b) $\lambda_1 \lambda_2$ complex (conjugate, right?) $\Rightarrow$ FOCUS
c) $\lambda_1 \lambda_2$ real same sign $\Rightarrow$ NODE
d). $\lambda_1 \lambda_2$ " opposite sign $\Rightarrow$ SADDLE



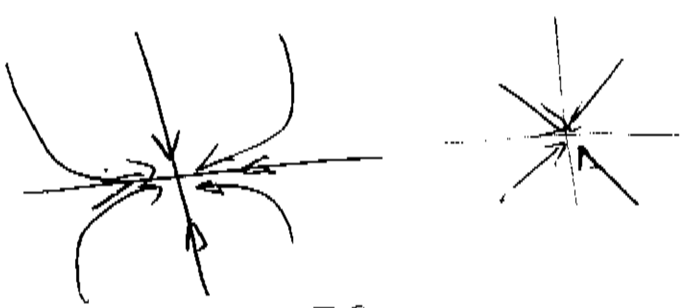
CENTER



FOCUS (stable)



FOCUS (Unstable)



NODES

SADDLE

⑥

a) $\lambda_1 \lambda_2$ imaginary. $\lambda_1 = i\omega_1$
$\lambda_2 = i\omega_2 = -i\omega_1$

$$\Rightarrow x = c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}$$
$$y = c_3 e^{i\omega_1 t} + c_4 e^{-i\omega_1 t}$$

$$\Rightarrow x = A \sin(\omega_1 t + \phi)$$
$$y = B \cos(\omega_1 t + \phi)$$

$$\boxed{\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1} \Rightarrow \underline{\text{elepse}}$$

b) Complex $\lambda_1 = a + i\omega_1$
$\lambda_2 = a - i\omega_1$

$$x = c_1 e^{i\omega_1 t} e^{at} + c_2 e^{at} e^{-i\omega_1 t}$$

$$\Rightarrow x = e^{at} A \sin(\omega t + \phi)$$
$$y = e^{at} B \cos(\omega t + \phi)$$

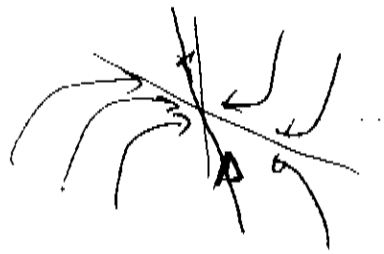
$$\boxed{e^{-2at}\left(\frac{x^2}{A^2} + \frac{y^2}{B^2}\right) = 1} \Rightarrow \text{FOCUS}$$

$\hookrightarrow$ elipses but larger (smaller) with time

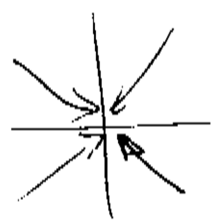
c) $\lambda_1\ \lambda_2$ real same sign

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_3 e^{\lambda_1 t} + c_3 e^{\lambda_2 t}$$

If $\lambda_1 \neq \lambda_2 \Rightarrow$ improper nodes

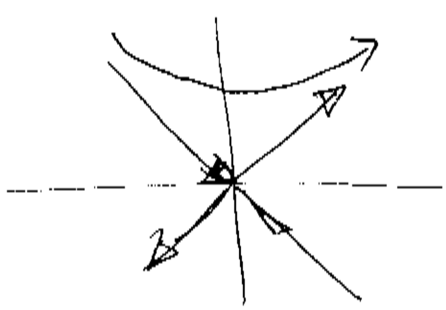


$\lambda_1 = \lambda_2$ proper node



d) $\lambda_1\ \lambda_2$ real opposite signes $\lambda_1 > 0$

$$x = c_1 e^{\lambda_1 t} + c_2 e^{-|\lambda_2| t}$$



Limit Cycles

Consider the Van der Pol equation (oscillations in a vacum tube circuits and the heart)

$$x'' - \varepsilon(1-x^2)x' + x = 0$$

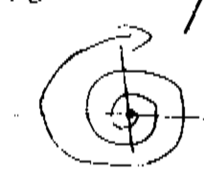
$$\Rightarrow \left.\begin{aligned} x' &= y \\ y' &= -x + \varepsilon(1-x^2)y \end{aligned}\right\} \quad \underbrace{\begin{aligned} y_s &= 0 \\ y_s &= \frac{x_s}{\varepsilon(1-x_s^2)} \end{aligned}}$$

$(y_s, x_s) = 0, 0)$

Linearize around $(0,0)$

$$\left.\begin{aligned} x' &= y \\ g' &= -x + \varepsilon y \end{aligned}\right| \quad \begin{aligned} &\varepsilon < 2 \text{ unstable face} \\ &\varepsilon > 2 \quad \text{"} \quad \text{node} \end{aligned}$$

$\to$ All trajectories go <u>away</u> from $(0,0)$



However as $|x| > 1$ then the sign of $\varepsilon(1-x^2)y$ changes introducing a damping which increases as the trajectory spirals out (it spends more time outside $|x| < 1$ than inside)

$\Rightarrow$ Does it stop at some orbit?