Approximate Solutions – Perturbation Methods (1)

Motivating example

solve $y^2 + y + \varepsilon = 0$ $\qquad \varepsilon$ small

let's ignore ε $\quad y^2 + y \sim 0 \quad \to \quad y = 0, \; y = -1$ solutions.

Real solution is

$$y(\varepsilon) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \varepsilon} \Rightarrow y(0) = \begin{cases} 0 \\ -1 \end{cases}$$

Expand $\sqrt{\frac{1}{4} - \varepsilon} = \frac{1}{2}\sqrt{1 - 4\varepsilon} = \frac{1}{2}\left(1 - 2\varepsilon - 2\varepsilon^2 \ldots\right)$

Taylor series

This suggest that we propose

$$y(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y_n$$

Substitute:

$$y^2 + y + \varepsilon = (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \ldots)^2 + (y_0 + \varepsilon y_1 + \ldots) + \varepsilon = 0$$

$$= (y_0^2 + 2 y_0 y_1 \varepsilon + (y_1^2 + 2 y_0 y_2)\varepsilon^2 \ldots) + $$

$$+ (y_0 + \varepsilon y_1 + \ldots) + \varepsilon = 0$$

$$\Rightarrow (y_0^2 + y_0) + (2 y_0 y_1 + y_1 + 1)\varepsilon + (y_1^2 + 2 y_0 y_2 + y_2)\varepsilon^2 + \ldots = 0$$

$$\Rightarrow \quad y_0^2 + y_0 = 0 \qquad \Rightarrow \quad y_0 = \begin{cases} 0 \\ -1 \end{cases}$$

$$2y_0y_1 + y_1 + 1 = 0$$

$$y_1^2 + 2y_0y_2 + y_2 = 0$$

$$2y_0y_1 + y_1 + 1 = 0 \begin{cases} \xrightarrow{y_0=0} & y_1 + 1 = 0 & y_1 = -1 \\ \xrightarrow{y_0=-1} & -y_1 + 1 = 0 & y_1 = 1 \end{cases}$$

$$y_1^2 + 2y_0y_2 + y_2 = 0 \begin{cases} \xrightarrow{y_0=0} & 1 + y_2 = 0 & y_2 = -1 \\ \xrightarrow{y_0=-1} & 1 - 2y_2 + y_2 = 0 & y_2 = 1 \end{cases}$$

This gives rise to two series

$$\left.\begin{aligned} y(\varepsilon) &= 0 - \varepsilon - \varepsilon^2 \dots \\ y(\varepsilon) &= -1 + \varepsilon + \varepsilon^2 \end{aligned}\right\}$$ which matches our Taylor expansion

ENCOURAGING!

Second Motivating example

$$y'' - \varepsilon y = 0$$
$$y'(0) = 0$$
$$y(1) = 1$$

$\Rightarrow$ Exact solution is

$$y = \frac{\cosh(\sqrt{\varepsilon}x)}{\cosh(\sqrt{\varepsilon})}$$

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Assume $y(x,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$

$$y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \dots = \varepsilon y_0 + \varepsilon^2 y_1 + \varepsilon^3 y_1 \dots$$

$$\left.\begin{array}{ll} y_0'(0) = 0 & y_i'(0) = 0 \\ y_0(1) = 1 & y_i(1) = 0 \end{array}\right\} \forall i \neq 0$$

$O(\varepsilon^0)$: $y_0'' = 0 \Rightarrow y_0 = ax + b$

use bc. $y_0'(0) = 0 \Rightarrow a = 0$

$y_0(1) = 1 \Rightarrow b = 1$

$$\boxed{y_0 = 1}$$

$O(\varepsilon)$: $y_1'' = y_0 \Rightarrow y_1'' = 1 \Rightarrow y_1' = x + a$

$\Rightarrow y_1 = \frac{x^2}{2} + ax + b$

use bc. $y_1'(0) = a = 0$

$y_1(1) = \frac{1}{2} + b = 0$

$\Rightarrow b = -\frac{1}{2}$

$$\boxed{y_1 = \frac{x^2}{2} - \frac{1}{2}}$$

$O(\varepsilon^2)$: $y_2'' = y_1 = \frac{x^2}{2} - \frac{1}{2} \Rightarrow y_2' = \frac{x^3}{6} - \frac{1}{2}x + a$

$y_2 = \frac{x^4}{24} - \frac{1}{4}x^2 + ax + b$

use bc $y_2'(0) = 0 \Rightarrow a = 0$

$y_2(1) = \frac{1}{24} - \frac{1}{4} + b = 0$

$\Rightarrow b = 5/24$

$$\Rightarrow \boxed{y_2 = \frac{5}{24} - \frac{x^2}{4} + \frac{x^4}{24}}$$

$$\Rightarrow y(x,\varepsilon) = 1 - \frac{\varepsilon}{2}(1-x^2) + \frac{\varepsilon^2}{4}\left(\frac{5}{6} - x^2 - \frac{x^4}{6}\right) + O(\varepsilon^3)$$

Now let's compare with the real solution.

$$\cosh(\varepsilon) = 1 + \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} + \cdots$$

but $(1+a\varepsilon)^{-1} = 1 - a\varepsilon + a^2\varepsilon^2 - a^3\varepsilon^3 \ldots$

$$\Rightarrow \frac{1}{\cosh(\sqrt{\varepsilon})} = 1 - \left(\frac{\varepsilon}{2!} + \frac{\varepsilon^2}{4!} + \cdots\right) + \left(\frac{\varepsilon}{2!} + \frac{\varepsilon^2}{4!} + \cdots\right)^2 + \cdots$$

$$\Rightarrow \frac{\cosh(\sqrt{\varepsilon}\,x)}{\cosh(\sqrt{\varepsilon})} = \left(1 + \frac{\varepsilon x^2}{2!} + \frac{\varepsilon^2 x^2}{4!} + \cdots\right)\left(\qquad\right)$$

$$= 1 - \frac{\varepsilon}{2}(1-x^2) + \frac{\varepsilon^2}{4}\left(\frac{5}{6} - x^2 + \frac{x^4}{6}\right) + O(\varepsilon^3)$$

Remarkable! (?)

WORKS! (At least in the above cases!)

Another example

$$y' = -\varepsilon y^2$$ (Second order reaction in a batch reactor)

IC $y(0) = 1$

Let's try $y(x, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t)$

$$y_0' + y_1'\varepsilon + \dots = -\varepsilon \left[y_0^2 + 2y_0y_1\varepsilon + (y_1^2 + 2y_0y_2)\varepsilon^2 + \dots \right]$$

$= 0$

IC $y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots = 1 \quad (t = 0)$

$O(\varepsilon^0)$: $\left. \begin{array}{l} y_0' = 0 \\ y_0(0) = 1 \end{array} \right\} \Rightarrow y_0 = 1$

$O(\varepsilon)$: $\left. \begin{array}{l} y_1' = -y_0^2 \\ y_1(0) = 0 \end{array} \right\} \Rightarrow y_1 = -t$

$O(\varepsilon^2)$: $\left. \begin{array}{l} y_2' = -2y_0y_1 \\ y_2(0) = 0 \end{array} \right\} \Rightarrow y_2 = t^2$

$\Rightarrow y = 1 - \varepsilon t + \varepsilon^2 t^2 - \varepsilon^3 t^3 + \dots$

which can be recognized as the expansion of

$$\boxed{y = \frac{1}{1+\varepsilon t}}$$

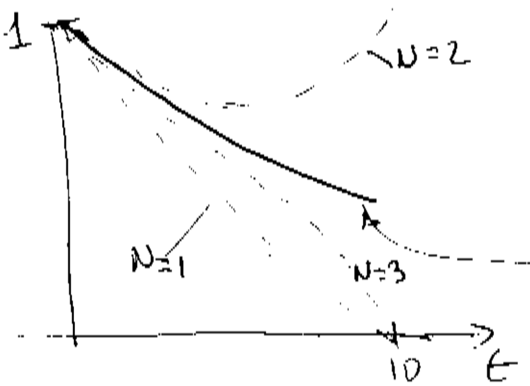
Verify

$$y' = \frac{-\varepsilon}{(1+\varepsilon t)^2} = -\varepsilon y^2 \quad ! \quad \underline{\text{WORKS !!}}$$

Let's go back to $y = 1 - \varepsilon t + \varepsilon^2 t^2 + \cdots$

we call $y_N = \sum_{n=0}^{N} \varepsilon^n y_n(t)$.

Let's look at the behaviour of these truncated series



There always exist t sufficiently large for which y_N does not approximate the solution well.

<u>OOPS</u> ! we are in trouble for the case of I.C. !

$\Rightarrow$ $\boxed{\text{Cannot work with } \infty \text{ domains this way !}}$

Another example

$$\left\{\begin{array}{l} \varepsilon y'' = y \\ y'(0) = 0 \\ y(1) = 1 \end{array}\right\} \text{ exact solution } \quad y = \frac{\cosh(x/\sqrt{\varepsilon})}{\cosh(1/\sqrt{\varepsilon})}$$

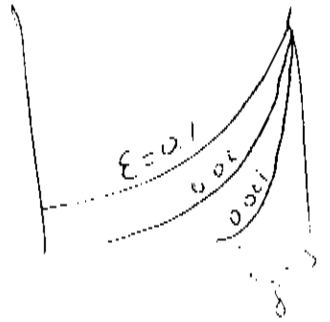
Try. $\quad y(x,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$

$$\left\{\begin{array}{l} \varepsilon(y_0'' + \varepsilon y_1'' + \cdots) = y_0 + \varepsilon y_1 \cdots \\ y_0'(0) + \varepsilon y_1'(0) + \cdots = 0 \quad \Rightarrow \cancel{\sum y_n'(0) = 0 \ \forall n} \\ y_0(1) + \varepsilon y_1(1) + \cdots = 1 \end{array}\right.$$

$\Rightarrow \quad y_0 = 0 \quad \Rightarrow \quad y_0'' = y_1 \quad \Rightarrow \quad y_1 = 0$

NO SOLUTION CAN BE OBTAINED !!

Let's look at the solution!



As $\varepsilon \to 0$

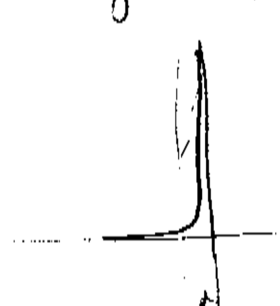
$$y \sim \exp\left[-\frac{(1-x)}{\sqrt{\varepsilon}}\right]$$

$$\delta = O(\sqrt{\varepsilon})$$

$\Rightarrow y = 0$ is, after all, correct away from

The above problem is called
SINGULAR PERTURBATION
PROBLEM

We notice that the solution close to $x=1$ is rapidly varying, while away from $x=1$ is zero



we call this a boundary layer.

What if we decide to stretch coordinates to capture the action.

Let $z = \frac{(1-x)}{\varepsilon^m}$ m undetermined.

ε^m will be the thickness of the BL

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{-1}{\varepsilon^m}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{1}{\varepsilon^m} = \frac{1}{\varepsilon^{2m}}\frac{d^2y}{dz^2}$$

$$\Rightarrow \varepsilon y'' = \frac{\varepsilon}{\varepsilon^{2m}}\frac{d^2y}{dz^2} = \varepsilon^{1-2m}\frac{d^2y}{dz^2}$$

$$\Rightarrow \text{New equation is } \varepsilon^{1-2m}\frac{d^2y}{dz^2} = y$$

Now $m = \frac{1}{2} \Rightarrow \frac{d^2y}{dz^2} = y$

$m = 1 \Rightarrow \frac{d^2y}{dz^2} = \varepsilon \cdot y$

$m = -\frac{1}{2} \Rightarrow \varepsilon^2 \frac{d^2y}{dz^2} = y$ $\Big\|$ ⇐ Takes us nowhere

For $m=1$ $\frac{d^2y_1}{dz^2} = \varepsilon y \Rightarrow y_0 = A + Bx$

$y_0(1) = 1 = A + B$

$\Rightarrow y_0 = A + (1-A)x$

Problem ⇒ does not tend to zero for large x.

Won't work -

$\Rightarrow$ Solve $\frac{d^2y}{dz^2} = y$, $y(1) = 1$ ⇐ $\left\{\right.$ We call this <u>INNER PROBLEM</u>

The solution is the INNER SOLUTION.

Notice we cannot use the other BC for this problem, (inner layer does not extend to zero).

In fact ∞ is the end of the BL for this problem.

$$\Rightarrow \quad y = A e^{z} + B e^{-z}$$

$x=1 \Rightarrow z=0 \Rightarrow$ $y(0) = A \cancel{e} + B \cancel{e}^{-1} = 1$

$$\Rightarrow A = \frac{1 - B/e}{e}$$

$$y = \left(1 - B/e\right) \frac{1}{e} e^{z} + B e^{-z}$$

As $z \to \pm\infty$ y is unbounded! -
but it needs to tend somehow to $y=0$ which we call the OUTER SOLUTION (of the OUTER PROBLEM).

We need a <u>MATCHING CONDITION</u>

means outer — means inner.

$$\lim_{x \to 1} y_0^o = \lim_{z \to \infty} y_0^i$$

Now $y_0^o = 0 \Rightarrow \lim_{x \to 1} y_0^o = 0$

But $\lim_{z \to \infty} y_0^i = \lim_{z \to \infty} \left\{ \left(1 - \frac{B}{e}\right)\frac{1}{e} e^{z} + B e^{-z} \right\}$

$= 0$

$\Rightarrow B = e$

$\Rightarrow y_0^i = e^{-z}$

=== o ===

COMPLETE SOLUTION

$$y^o(x) = y_0^o(x) + y_0^i(x) - (\text{common part})$$

$$= 0 + e^{-z} - 0$$ in this case.

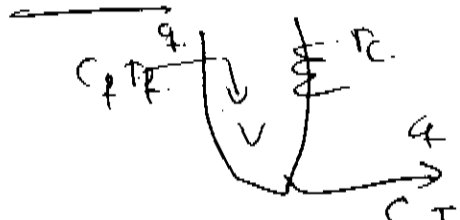
recall $m = 1/2$

$$y^o(x) = e^{-\left(\frac{1-x}{\varepsilon^{1/2}}\right)}$$

Where do the ε come from

Two examples

CSTR



$$V\frac{dc}{dt} = q(C_f - C) - V\,f(C,T)$$

$$V c_p \rho \frac{dT}{dt} = q\, c_p \rho (T_f - T) - Ua(T - T_c) + (-\Delta H) V\, f(C,T)$$

$$C(0) = C_0$$
$$T(0) = T_0$$

Nondimensionalize: $u = \frac{C}{C_f}$, $v = \frac{T}{T_f}$

$$r(u,v) = \frac{f(C,T)}{f(C_f,T_f)}$$

Time is a little more tricky. You need something called "characteristic time". In this case it is the residence time

$\theta = V/q \Rightarrow \tau = t/\theta$

$$\Rightarrow \frac{du}{d\tau} = 1 - u - \phi\, r(u,v)$$
$$\frac{dv}{d\tau} = 1 - v - \delta(v - v_c) + \beta\phi\, r(u,v)$$

$$\phi = \theta \frac{f(C_f, T_f)}{C_f} \qquad \beta = \frac{(-\Delta H) C_f}{c_p \rho T_f} \qquad \delta = \frac{Ua}{q c_p \rho}$$

ϕ, β, δ are parameters which can have low or large values.
For example if ϕ is small a regular perturbation solution is possible
When ϕ is large then let $\varepsilon = 1/\phi$ and

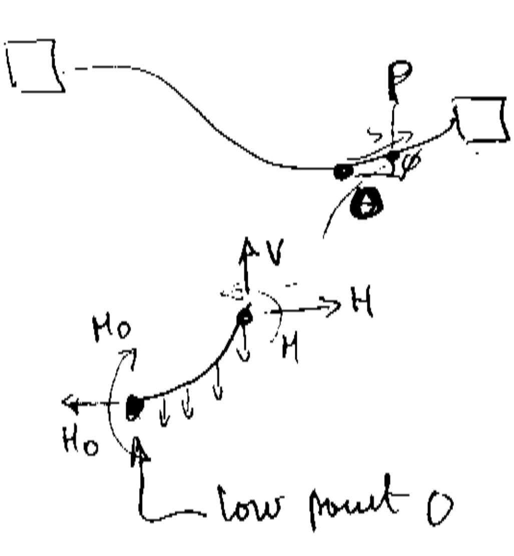
$$\varepsilon \frac{du}{d\tau} = \varepsilon(1-u) - r(u,v)$$

$$\varepsilon \frac{dv}{d\tau} = \varepsilon\left[1 - v - \delta(v - v_c)\right] + \beta r(u,v)$$

and a singular perturbation scheme can be applied!

Hanging Cable Problem

Consider a cable



s - distance from low point

Forces Balance

$$H(s) = H(0) = H_0$$

$$V(s) = \int_0^s w(t)\,dt$$

Moment Balance

$$M(s) = M_0 - x(s)V(s) + y(s)H(s) + \int_0^s x(t)\,w(t)\,dt$$

But

$$M(s) = EI(s)\frac{d\phi}{ds}$$

$$\Rightarrow (EI\phi')' = M' = -x'\int_0^s w(t)\,dt \cancel{- x w} + y' H_0 \cancel{+ x w}$$

$$\Rightarrow \begin{cases} (EI\phi')' + \cos\phi \displaystyle\int_0^s w(t)\,dt - H_0 \sin\phi = 0 \\ \phi(0) = \phi(L) = 0 \end{cases}$$

Let $z = s/L$, $k = \dfrac{wL}{H_0}$, $\varepsilon^2 = \dfrac{EI}{L^2 H_0}$

$$\Rightarrow \begin{cases} \varepsilon^2 \phi'' + k z \cos\phi - \sin\phi = 0 \\ \phi(0) = \phi(1) = 0 \end{cases}$$

Now $\phi'' = \dfrac{d^2\phi}{dz^2}$

ε small (bending stiffness small)

$\Rightarrow$ singular problem.

LOCATION OF BOUNDARY LAYER

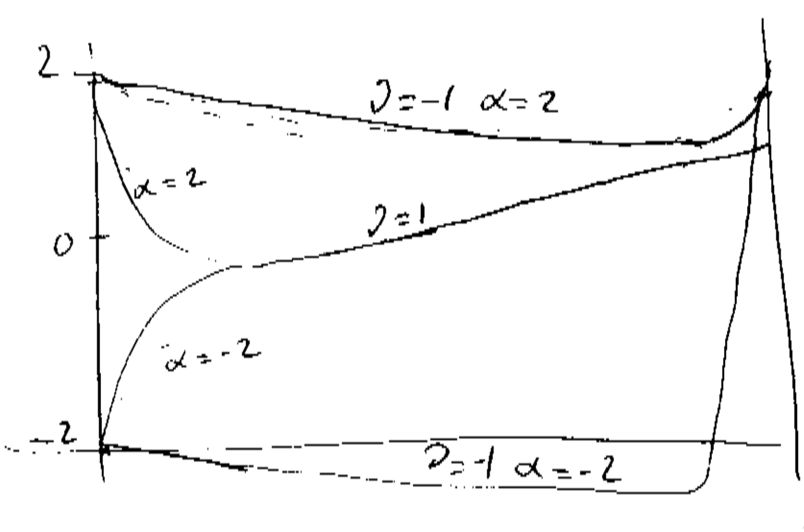
Let us solve

$$\begin{cases} \varepsilon y'' + \nu y' = e^x \\ y(0) = \alpha \\ y(1) = 1 \end{cases} \qquad \nu = \pm 1$$

Solution is $\quad y = C_1 + C_2\, e^{-\nu x/\varepsilon} + \dfrac{e^x}{\nu+\varepsilon}$

$$C_1 = \frac{[\alpha(\nu+\varepsilon)-1]\,e^{-\nu/\varepsilon} + e - (\nu+\varepsilon)}{(\nu+\varepsilon)(e^{-\nu/\varepsilon}-1)}$$

$$C_2 = \frac{(1-\alpha)(\nu+\varepsilon) + 1 - e}{(\nu+\varepsilon)(e^{-\nu/\varepsilon}-1)}$$



$\varepsilon = 0.01$

Unclear where the BL will develop.

Try all possible locations of BL.

BL at $x=0$

Outer expansion
$$\left.\begin{aligned} \nu \frac{dy_0^o}{dx} &= e^x \\ y_0^o(0) &= 1 \end{aligned}\right\}$$

$$\Rightarrow y_0^o(x) = 1 + \frac{1}{\nu}\left(e^x - e\right)$$

And $\lim_{x\to 0} y_0^o = 1 + \frac{(1-e)}{\nu}$

Inner expansion

$$z = \frac{x}{\varepsilon^m}$$

$$\frac{d^2y}{dz^2} + \nu \cdot \varepsilon^{m-1}\frac{dy}{dz} = \varepsilon^{2m-1}\exp(s\varepsilon^m)$$

$m > 1$:
$$\left.\begin{aligned} \frac{d^2y_0^i}{dz^2} &= 0 \\ y_0^i(0) &= \alpha \end{aligned}\right\} \quad y_0^i = \alpha + cz$$

Matching?

$$\lim_{z\to\infty} y_0^i(z) = \lim_{x\to 0} y_0^o(x) = 1 + \left(\frac{1-e}{\nu}\right)$$

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$\Rightarrow \quad C = 0$

$\Rightarrow \quad \alpha = 1 + (1 - \frac{e}{\nu})$ contradiction!

$m = 1.$

$$\left.\begin{array}{l} \dfrac{d^2 y_0^i}{dz^2} + \nu \dfrac{d y_0^i}{dz} = 0 \\ y_0^i(0) = \alpha \end{array}\right\} \Rightarrow y_0^i(z) = \alpha - C(1 - e^{-\nu z})$$

Matching!

$$\lim_{z \to \infty} [\alpha - C(1 - e^{-\nu z})] = \lim_{x \to 0} y_0^0 = 1 + (1 - \frac{e}{\nu})$$

$\Rightarrow C = \alpha - 2 + e \qquad (\nu = +1)$

$C = 0 \qquad \nu = -1$

$\Rightarrow$ we are OK if $\nu = +1$

we are OK so far for $\nu = -1$ only if $\alpha = e$

$0 < m < 1$

$$\left.\begin{array}{l} \dfrac{d y_0^i}{dz} = 0 \\ y_0^i(0) = \alpha \end{array}\right| \Rightarrow \underline{y_0^i = \alpha}$$

NOT GOOD!

If you try BC at $x=1$ you will find that

$$\boxed{z = \frac{1-x}{\varepsilon^m}}$$

$$\frac{d^2 y^i}{dz^2} - \nu \varepsilon^{m-1} \frac{dy^i}{dz} = e\,\varepsilon^{2m-1}\left[1 - \varepsilon^m z + \cdots\right.$$

$$y^i(0) = 1$$

$x=1$

We have again 3 cases.

$m=1$ is the only one that makes sense again.

$$y_0^i = 1 - C\left(1 - e^{\nu z}\right)$$

$$\lim_{z \to \infty} \left[1 - C\left(1 - e^{\nu z}\right)\right] = \lim_{x \to 1} \left[\alpha - \frac{1}{\nu}\left(1 - e^{x}\right)\right]$$

$$\Rightarrow C = e - \alpha \qquad (\nu = -1)$$

— o —

Let's formalize the

MATCHING PRINCIPLE

$$\lim_{\varepsilon \to 0} \left\{ \begin{array}{l} m\text{-term outer expansion} \\ \text{expressed in inner} \\ \text{variables} \end{array} \right\} = \lim_{\varepsilon \to 0} \left\{ \begin{array}{l} k\text{-term inner} \\ \text{expansion} \\ \text{expressed} \\ \text{in outer} \\ \text{coordinates} \end{array} \right\}$$

Illustration

$k = m = 1 \qquad D = +1$

$$y^o = \underbrace{(e^x + 1 - e) + \varepsilon\,(e - e^x)}_{\text{Two term outer expansion}} + O(\varepsilon^2)$$

$$y^i = \underbrace{[\alpha - C_1(1 - e^{-z})] + \varepsilon\,[C_2(1 - e^{-z}) + z]}_{\text{Two term inner expansion}} + O(\varepsilon$$

$k = m = 1$

1-term outer solution $y^o_0 = e^x + 1 - e$

$m=1$ It's 1-term inner expansion with $x = \varepsilon z$ is

$$\lim_{\substack{\varepsilon \to 0 \\ z \text{ fixed}}} [e^{\varepsilon z} + 1 - e] = 2 - e$$

We are expanding outer solution using inner variables.

1 term inner solution is $y_0^i = \alpha - c_1(1-e^{-z})$
In outer coordinates is $y_0^i = \alpha - c_1(1-e^{-x/\varepsilon})$

$$\lim_{\substack{\varepsilon \to 0 \\ z \text{ fixed}}} \left\{ e^{\varepsilon z} + 1 - e \right\} = 2 - e = \lim_{\substack{\varepsilon \to 0 \\ x \text{ fixed}}} \left\{ \alpha - c_1(1-e^{-x/\varepsilon}) \right\}$$

$$= \alpha - c_1$$

$$\Rightarrow \boxed{c_1 = \alpha + e - 2}$$

$\boxed{k=2, \; m=2}$

$(1 + \varepsilon z + \frac{\varepsilon^2 z^2}{2!} \ldots)(\quad)$
$= (1-\varepsilon) + \varepsilon z + O(\varepsilon^2)$

$$\lim_{\substack{\varepsilon \to 0 \\ z \text{ fixed}}} \left\{ e^{\varepsilon z} + 1 - e + \varepsilon(e - e^{\varepsilon z}) \right\} = (2-e) + \varepsilon(\cancel{z} + e - 1)$$
$$= (2 - e + x) + \varepsilon(e-1)$$

$$\lim_{\substack{\varepsilon \to 0 \\ x \text{ fixed}}} \left\{ \alpha - c_1(1-e^{-x/\varepsilon}) + \varepsilon\left[c_2(1-e^{-x/\varepsilon}) + \frac{x}{\varepsilon} \right] \right\} =$$
$$= (\alpha - c_1 + x) + \varepsilon c_2$$

$$\Rightarrow \boxed{\begin{aligned} c_1 &= \alpha + e - 2 \\ c_2 &= e - 1 \end{aligned}}$$

Composite Solution

$$y^c(x) = y^0(x) + y^i(x) - \text{common part}$$

$$y^0(x) = (e^x + 1 - e) + \varepsilon(e - e^x) + O(\varepsilon^2)$$

$$y^i(x) = (\alpha - (\alpha + e - 2)(1 - e^{-z}) + \varepsilon[(e-1)(1-e^{-z}) + z] + O(\varepsilon^2)$$

$$= (2 - e) - (\alpha + e - 2)e^{-x/\varepsilon} + \varepsilon[(e-1)(1 - e^{-x/\varepsilon}) + \tfrac{x}{\varepsilon}] + O(\varepsilon^2)$$

Common part is the one obtain in the limits (previous page).

$$\text{common part} = (2-e) + \varepsilon\left(\frac{x}{\varepsilon} + e - 1\right)$$

$$= (2 - e + x) + \varepsilon(e-1)$$

$\Rightarrow$

$$y^c(x) = (e^x + 1 - e) + (\alpha - 2 + e)e^{-x/\varepsilon} + \varepsilon[(e - e^x) - (e-1)e^{-x/\varepsilon}] + O(\varepsilon^2)$$

Internal Boundary layer

$$\begin{cases} \varepsilon(x+2)\, y'' + x\, y' - x^2 y = 0 \\ y(-1) = -2 \\ y(1) = 1 \end{cases}$$

Outer solution

$$y^o = \sum_{n=0}^{\infty} \varepsilon^n y_n^{(o)}(x)$$

$$\Rightarrow x\, y_0^{o\,\prime} - x^2 y_0^o = 0$$

We assume one BC only at a time. Why?

$$y_0^o(-1) = 2 \Rightarrow y_0^o(x) = -2\, e^{-\frac{(1-x^2)}{2}}$$

$$y_0^o(1) = 1 \Rightarrow y_0^o(x) = e^{-\frac{(1-x^2)}{2}}$$

we do not know where the BL is so we keep both outer solutions for the time being.

Inner solution

at $x = -1$

$$\Rightarrow z = \frac{(x+1)}{\varepsilon^m}$$

$$\Rightarrow \varepsilon^{1-2m}(z\varepsilon^m+1)\frac{d^2y}{dz^2} + \frac{(z\varepsilon^m-1)}{\varepsilon^m}\frac{dy}{dz} - (z\varepsilon^m-1)^2 y = 0$$

dominated by

Need to balance these

$\Rightarrow$

$$\varepsilon^{1-m}(z\varepsilon^m+1)\frac{d^2y}{dz^2} + (z\varepsilon^m-1)\frac{dy}{dz} - \varepsilon^m(z\varepsilon^m-1)^2 y = 0$$

pick $m = 1$

$$\frac{d^2y_0^i(z)}{dz^2} - \frac{dy_0^i}{dz} = 0 \qquad \text{let } R = \frac{dy_0^i}{dz}$$

$$R_z - R = 0$$

$$\frac{dR}{R} = dz \Rightarrow R = Ae^z$$

$$\Rightarrow y_0^i(z) = B + Ce^z$$

Need $y_0^i(0) = -2$ $\Rightarrow B + C = -2$

$x = -1$

$\Rightarrow B = -2 - C$

$$\Rightarrow \boxed{y_0^i(z) = -2 + C\left(e^{z} - 1\right)}$$

Need to Match with outer using BC at $x = 1$

$$\lim_{\substack{\varepsilon \to 0 \\ x \text{ fixed}}} y_0^i\left(\frac{x+1}{\varepsilon}\right) = \lim_{\substack{\varepsilon \to 0 \\ z \text{ fixed}}} y_0^o(z\varepsilon - 1)$$

$$\lim_{\substack{\varepsilon \to 0 \\ x \text{ fixed}}} \left[-2 + C\left[e^{\frac{(x+1)}{\varepsilon}} - 1\right]\right] = \lim_{\substack{\varepsilon \to 0 \\ z \text{ fixed}}} e^{-\frac{1}{2}} e^{\frac{(z\varepsilon - 1)^2}{2}} = 1$$

Trouble $\to \infty$

$\Rightarrow C = 0$

$\Rightarrow -2 = 1$ Contradiction!!

$\Rightarrow$ There is the same trouble for $x = 1$.

$\Rightarrow$ BL is somewhere *else*!

Try. $z = \dfrac{(x-a)}{\varepsilon^{m}}$

$$\varepsilon^{1-2m}\left(z\varepsilon^{m} + a + 2\right)\frac{d^2y}{dz^2} + \frac{(z\varepsilon^{m}+a)}{\varepsilon^{m}}\frac{d^2y}{dz^2} - (z\varepsilon^{m}+a)\,y = 0$$

$\Rightarrow$ $m=1$ $\Rightarrow$ $\dfrac{d^2y_0^i(z)}{z^2} - \dfrac{dy_0^i}{dz} = 0$

As before

$\Rightarrow$ $y_0^i(z) = B + C e^{z}$

For the region $[-1, a]$ we have

$$\lim_{\varepsilon\to 0} y_0^i\left(\frac{x-a}{\varepsilon}\right) = \lim_{\varepsilon\to 0} y_0^o(\varepsilon z + a)$$

$$\lim_{\varepsilon\to 0}\left(B + C e^{\frac{x-a}{\varepsilon}}\right) = B = \lim_{\varepsilon\to 0}\left\{-2\, e^{-\left\{\frac{1-(\varepsilon z+a)^2}{2}\right\}}\right\}$$

$$= -2\, e^{-\frac{(1-a^2)}{2}}$$

For the region $[a, 1]$

$$\lim_{\varepsilon \to 0} y_0^i\left(\frac{x-a}{\varepsilon}\right) = \lim_{\varepsilon \to 0}\left(B + C e^{\frac{x-a}{\varepsilon}}\right)$$

$\to \infty$

$\Rightarrow C = 0$

$$\Rightarrow B = \lim_{\varepsilon \to 0}\left\{ e^{-\left[\frac{1-(\varepsilon z+a)^2}{2}\right]}\right\}$$

$$\Rightarrow B = e^{-\left[\frac{1-a^2}{2}\right]} \quad \text{contradiction}$$

$\Rightarrow$ Try $a = 0$.

$$\Rightarrow \varepsilon^{1-2m}\left(z\varepsilon^{m} + 2\right)\frac{d^2y}{dz^2} + z\frac{dy}{dz} - z^2\varepsilon^{2m} y = 0$$

again: need to balance $\Rightarrow m = 1/2$.

$$\Rightarrow 2y_{zz} + z y_z = 0$$

$$y_z = A\, e^{-z^2/4} \Rightarrow y_0^i(z) = y_0^i(0) + A\int_0^z e^{-t^2/4}\, dt$$

$$\Rightarrow y_0^i(z) = y_0^i(0) + B\,\mathrm{erf}(z/2)$$

Since $m = \frac{1}{2}$ our expansion will be

$$\begin{cases} y^{o}(x,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{n/2} y_n^{o}(x) \\ y^{i}(\ ) = \sum_{n=0}^{\infty} \varepsilon^{n/2} y_n^{i}(x) \end{cases}$$

the rest is standard.

Need Matching for both sides!

Method of Strained Coordinates (for removing secular terms)

Consider

$$y'' + y - \varepsilon y^3 = 0 \qquad t > 0$$

$$y(0) = a$$

$$y'(0) = 0$$

(This is the spring equation εy^3 is a small deviation from Hook's law)

Try $y = \sum \varepsilon^n y_n(t)$

$$\left\{\begin{array}{l} (y_0'' + \varepsilon y_1'' + \cdots) + (y_0 + \varepsilon y_1 + \cdots) - \varepsilon(y_0 + \varepsilon y_1 \cdots)^3 = 0 \\ y_0(0) = a \\ y_i(0) = 0 \quad \forall i \neq 0 \\ y_i'(0) = 0 \quad \forall i \end{array}\right.$$

$$\left.\begin{array}{l} y_0'' + y_0 = 0 \\ y_0(0) = a \\ y_0'(0) = 0 \end{array}\right\} \Rightarrow \quad y_0 = a \cos t$$

$$\left.\begin{array}{l} y_1'' + y_1 = a^3 \cos^3 t = \frac{a^3}{4}(\cos 3t + 3\cos t) \\ y_1(0) = 0 \\ y_1'(0) = 0 \end{array}\right\}$$

$$y_1 = \frac{a^3}{32}(\cos t - \cos 3t) + \frac{3a^3}{8} t \sin t$$

$\Rightarrow$

$$y = a\cos t + \varepsilon \frac{a^3}{32}\left[\cos t - \cos 3t + 12\, t \sin t\right] + O(\varepsilon^2)$$

y does not converge uniformly in $(0, \infty)$
The term $\underline{t \sin t}$ grows unboundedly.

We use strained coordinates to remove secular terms like $t \sin t$ -

Let
$$y = \sum_{n=0}^{\infty} y_n(\tau)\, \varepsilon^n$$
$$t = \tau + \varepsilon f_1(\tau) + \varepsilon^2 f_2(\tau) + \cdots$$

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt}\ , \quad \frac{d^2y}{dt^2} = \frac{d^2y}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 + \frac{dy}{d\tau}\frac{d^2\tau}{dt^2}$$

$$\Rightarrow \frac{dy}{d\tau} = \sum_{n=0}^{\infty} y_n'(\tau)\, \varepsilon^n$$

$$\frac{dt}{d\tau} = 1 + \varepsilon \frac{df_1}{d\tau} + \varepsilon^2 \frac{df_2}{d\tau^2}$$

$$\Rightarrow \frac{d\tau}{dt} = \frac{1}{1 + \sum_{n=1}^{\infty} \varepsilon^n \frac{df_n}{d\tau}} = 1 - \varepsilon \frac{df_1}{d\tau} + \varepsilon^2 \frac{df_2}{d\tau^2} + \cdots$$

$$\Rightarrow \frac{dy}{dt} = \left(\frac{dy_0}{d\tau} + \varepsilon \frac{dy_1}{d\tau} + \cdots\right)\left(1 - \varepsilon \frac{df_1}{d\tau} + \cdots\right)$$

Similarly

$$\frac{d^2y}{dt^2} = \left[\left(\frac{d^2y_0}{d\tau^2} + \varepsilon\frac{d^2y_1}{d\tau^2} + \cdots\right)\left(1 - \varepsilon\frac{df_1}{d\tau} + \cdots\right)^2 + \left(\frac{dy_0}{d\tau} + \varepsilon\frac{dy_1}{d\tau} + \cdots\right)\left(-\varepsilon\frac{d^2f_1}{d\tau^2} + \cdots\right)\right]$$

$$\Rightarrow \quad = \frac{d^2y_0}{d\tau^2} + \varepsilon\left[\frac{d^2y_1}{d\tau^2} - 2\varepsilon\frac{df_1}{d\tau}\frac{dy_0^2}{d\tau^2} - \frac{d^2f_1}{d\tau^2}\frac{dy_0}{d\tau}\right] + O(\varepsilon^2)$$

$$\Rightarrow \frac{d^2y_0}{d\tau^2} + \varepsilon\left[\frac{d^2y_1}{d\tau^2} - 2\varepsilon f_1'\frac{d^2y_0}{d\tau^2} - f_1''\frac{dy_0}{d\tau}\right] + $$

$$+ \left[y_0 + \varepsilon y_1 + \cdots\right] - \varepsilon\left(y_0^3 + 3\varepsilon y_0^2 y_1 + \cdots\right) = 0$$

Since we want $t = 0 \Rightarrow \tau = 0$

we set $f_n(0) = 0$ as a condition

$\Rightarrow$ I.C reduce to

$$y_0(0) + \varepsilon y_1(0) + \cdots = a$$

$$\frac{dy_0(0)}{d\tau} + \varepsilon\left[\frac{dy_1(0)}{d\tau} - f_1'(0)\frac{dy_0}{d\tau}(0)\right] + \cdots = 0$$

$$\Longrightarrow \left.\begin{aligned} &\frac{d^2y_0}{d\tau^2} + y_0 = 0 \\ &y_0(0) = a \\ &\frac{dy_0}{d\tau}(0) = 0 \end{aligned}\right\} \Rightarrow y_0 = a\cos\tau$$

$$\left\{\begin{aligned} &\frac{d^2y_1}{d\tau^2} + y_1 = 2f_1'\frac{d^2y_0}{d\tau^2} + f_1''\frac{dy_0}{d\tau} + y_0^3 \\ &\qquad = \frac{a^3}{4}\cos 3\tau - \underbrace{a\left(f_1''\sin\tau + 2f_1'\cos\tau - \frac{3a^2}{4}\cos\tau\right)} \\ &y_1(0) = 0 \\ &\frac{dy_1}{d\tau}(0) + a\,\frac{df_1(0)}{d\tau}\sin\tau = 0 \end{aligned}\right.$$

This will introduce secular terms
Make it zero

$$f_1''\sin\tau + 2f_1'\cos\tau = \frac{3a^2}{4}\cos\tau$$

$$\left.\begin{aligned} &\Rightarrow \frac{d}{d\tau}\left[\sin^2\tau\,\frac{df_1}{d\tau}\right] = \frac{3a^2}{4}\sin\tau\cos\tau \\ &\quad f_1(0) = 0 \end{aligned}\right\}$$

$$\Longrightarrow \boxed{f_1 = \frac{3a^2}{8}\tau}$$

$$\Rightarrow \left.\begin{aligned} &\frac{d^2 y_1}{d\tau^2} + y_1 = \frac{a^3}{4}\cos 3\tau \\ &y_1(0) = 0 \\ &\dot{y}_1(0) = 0 \end{aligned}\right\}$$

$$y_1(\tau) = c_1 \sin\tau + c_2 \cos\tau - \frac{a^3}{32}\cos 3\tau$$

$$\Rightarrow y_1(\tau) = \frac{a^3}{32}(\cos\tau - \cos 3\tau)$$

$$\Rightarrow y(\tau) = a\cos\tau + \varepsilon\frac{a^3}{32}(\cos\tau - \cos 3\tau)$$

But $t = \tau + \frac{3a^2}{8}\tau\,\varepsilon + \cdots$

$$\Rightarrow \tau = \frac{t}{1 + \varepsilon(\frac{3a^2}{8}) + \cdots} = \left(1 - \varepsilon\frac{3a^2}{8} + \cdot\right)t$$
$$= \omega(\varepsilon, a)\, t$$

which shows that the frequency depends on ε.

Where is the asymptotic expansion valid? (32)

All we know is that the solution is periodic

Let $y_1 = y$, $y_2 = \frac{dy_1}{dt}$ $\Rightarrow$

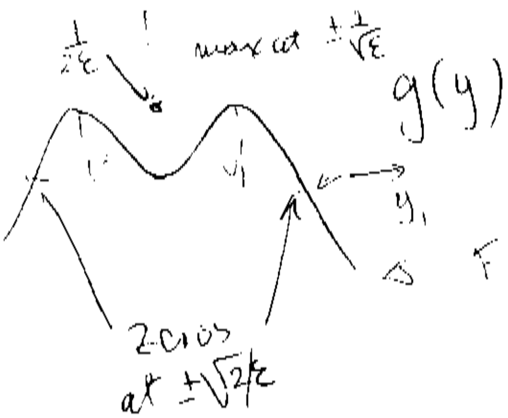
$$\frac{dy_1}{dt} = y_2$$

$$\frac{dy_2}{dt} = -y_1 + \epsilon y_1^3$$

Divide $\frac{dy_2}{dy_1} = \frac{-y_1 + \epsilon y_1^3}{y_2}$

$\Rightarrow$ integrate

$$y_2^2 = C - y_1^2 + \frac{\epsilon}{2} y_1^4 = C - g(y_1)$$



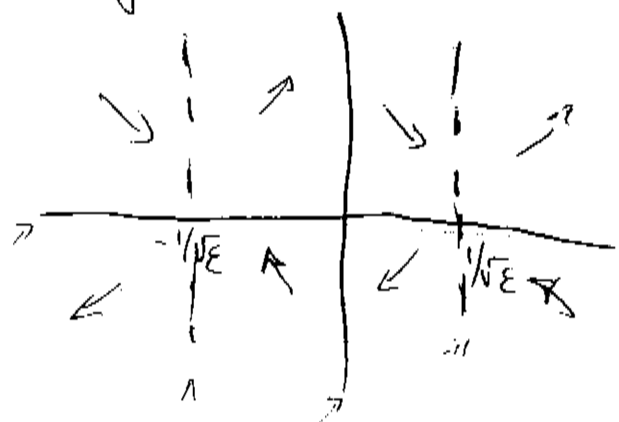
$$g(y_1) = y_1^2\left(1 - \frac{\epsilon}{2} y_1^2\right)$$

$$C = a^2 - \frac{\epsilon}{2} a^4$$

For large y_1 we can write

$$g(y_1) \sim -\frac{\epsilon}{2} y_1^4 \quad (\text{large } y_1)$$

Trajectories



$\frac{dy_1}{dt} = 0$ on this line

$\frac{dy_2}{dt} = 0$ on these lines

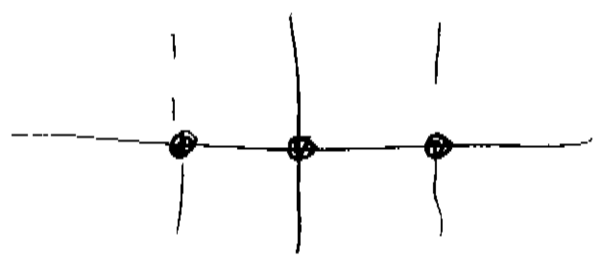
$\frac{dy_1}{dt} > 0 \quad y_2 > 0$

$\frac{dy_1}{dt} < 0 \quad y_2 < 0$

$\frac{dy_2}{dt} > 0 \quad y > \frac{1}{\sqrt{\varepsilon}}$

$\frac{dy_2}{dt} < 0 \quad y < -\frac{1}{\sqrt{\varepsilon}}$

$\Rightarrow$ There are 3 stationary points (steady states!)



Linearize around each of these points

$$\left.\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -y_1 + \varepsilon y_1^3 \end{aligned}\right) \Longrightarrow \quad \dot{y} = A y$$

$$A = \begin{bmatrix} 0 & 1 \\ -1+3\varepsilon y_1^2 & 0 \end{bmatrix}$$

Linearize

$$\left.\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &+\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i \end{aligned}\right\} \Rightarrow (0,0) \text{ is a center}$$

$(\pm \frac{1}{\sqrt{\varepsilon}}, 0)$ are both saddle

$$\begin{cases} (x_s, y_s) = (\frac{1}{\sqrt{\varepsilon}}, 0) \\ \text{let } Y_1 = y_1 - 1/\sqrt{\varepsilon}, \quad Y_2 = y_2 \end{cases} \quad \frac{dY_1}{dt} = Y_2, \quad \frac{dY_2}{dt} = Y_1 4 \quad A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \quad \lambda_{1,2} = \text{real}.$$

Let us see what are the trajectories depending on C.

1) $C < 0$

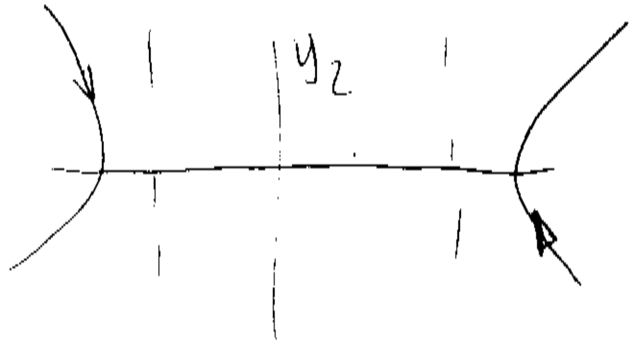
$\Rightarrow$ 

$C - g(y_1)$

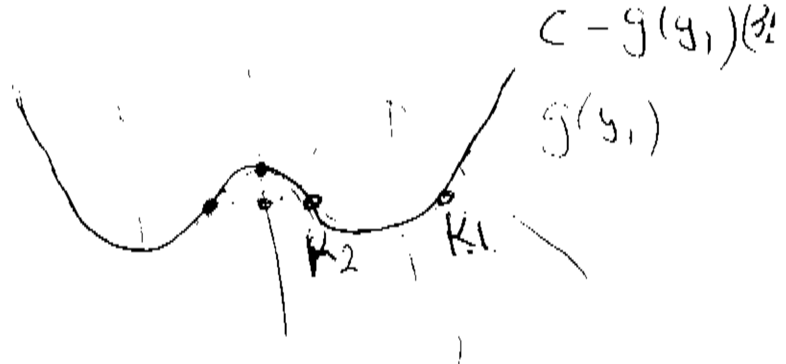
$\Rightarrow C - g(y_1) > 0$ for $y_1 \geq |K|$

$K \Rightarrow C = g(K)$

Since $y_2 = \sqrt{C - g(y_1)}$ $\Rightarrow$ trajectories exist (y_2 is real) only for $y_1 \geq |K|$



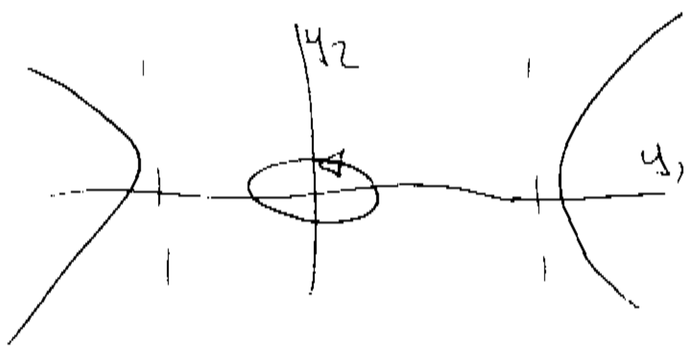
2) $0 < C < \frac{1}{2}\varepsilon$



$C - g(y_1) > 0$

$\sqrt{1/\varepsilon} < K_1 < \sqrt{2/\varepsilon}$

$0 < K_2 < \frac{1}{\sqrt{\varepsilon}}$



3) $C > \frac{1}{2}\varepsilon$