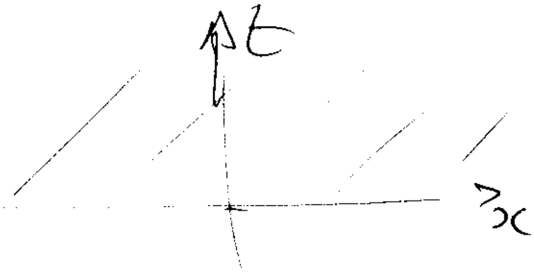
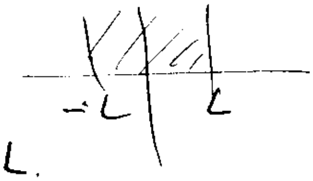
PDE on Infinite domains

Consider the heat conduction in an infinite rod.



$$\begin{cases} \alpha^2 u_{xx} = u_t \\ u(x,0) = f(x) \end{cases}$$

Consider the finite rod problem



We needed BC at L and -L.

$\Rightarrow$ Assume $L \to \infty$, then you will need some BC at ∞. We will come back to this.

Take a fourier transform.

$$F(\alpha^2 u_{xx} - u_t) = 0$$

$$\alpha^2 F(u_{xx}) - \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\omega x} dx = 0$$

$$\alpha^2 (i\omega)^2 \hat{u} = \frac{d}{dt} \int_{-\infty}^{\infty} u e^{-i\omega x} dx$$

$$= \frac{d\hat{u}}{dt}$$

$$\hat{u}' + \alpha^2 \omega^2 \hat{u} = 0$$

$$\Rightarrow \hat{u} = A e^{-\alpha^2 \omega^2 t}$$

From. $F(u_{xx}) = (i\omega)^2 \hat{u}$ we need

$$\begin{cases} u \to 0 & \text{as } x \to \pm\infty \\ u_x \to 0 \end{cases}$$

Therefore we adopt $\begin{matrix} u = 0 \\ u_x = 0 \end{matrix}$ at $x \to \pm\infty$ as BC. Incidentally, this restricts $f(x)$ to behave in the same way

Let us evaluate A

$$\hat{u}\Big|_{t=0} = \hat{f}(\omega) = (A e^{-\alpha^2\omega^2 t}\Big|_{t=0} = A$$

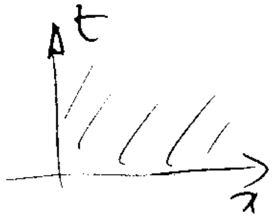
$$\Rightarrow \hat{u}(\omega, t) = \hat{f}(\omega) e^{-\alpha^2 \omega^2 t}$$

use convolution

$$u(x,t) = f(x) * \frac{1}{2\alpha\sqrt{\pi t}} e^{-x^2/4\alpha^2 t}$$

$$\Rightarrow u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-(x-\xi)^2/4\alpha^2 t}\, d\xi$$

Semi-infinite domains



$$\left\{\alpha^2 u_{xx} - u_t\right\} = 0 \qquad \begin{aligned} u(x,0) &= 0 \\ u(0,t) &= g(t) \end{aligned}$$

↑ which transform is appropriate

Answer: LAPLACE ; why? (left as an exercise)

More on this in the homework.