ODE

Linear equation (FIRST ORDER)

$$a_0(x)\,y' + a_1(x)\,y = f(x)$$

$$\Downarrow$$

$$y' + p(x)\,y = q(x)$$

Homogeneous case $q(x) = 0$

$$y' + p(x)\,y = 0$$

$$\frac{d\ln y}{dx} + p(x) = 0 \quad \Rightarrow$$

$$y = A e^{-\int p(x)\,dx}$$

Initial conditions (?)

$$y(a) = b$$

$$\Rightarrow \boxed{y = b\, e^{-\int_a^x p(\xi)\,d\xi}}$$

Nonhomogeneous Case

Integrating factor $\sigma(x)$

$$y' + p(x)\, y = q(x)$$

$$\Downarrow$$

$$\underbrace{\sigma y' + \sigma p y}_{} = \sigma q$$

$\Downarrow$ Choose $\sigma \ni$

$$\sigma y' + \sigma p y = \frac{d(\sigma y)}{dx} \implies \boxed{\sigma' = p\sigma}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\frac{d(\sigma y)}{dx} = \sigma q \qquad\qquad \sigma = e^{\int p\, d\xi}$$

$$\Downarrow$$

$$\sigma y = \int \sigma q\, dx + C$$

$$\Downarrow$$

$$y = \sigma^{-1}\left[\int \sigma q\, dx + C\right]$$

$$\Downarrow$$

$$y = e^{-\int p(\xi)\, d\xi}\left[\int e^{\int p(\xi)\, d\xi} q(\xi)\, d\xi + C\right]$$

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Is the solution unique? . Suppose not

$$\left.\begin{array}{l} y_1' + p y_1 = q \\ y_2' + p y_2 = q \end{array}\right\} \Rightarrow$$

continue from here

Variation of parameters approach

(Lagrange)

Start with the homogeneous sol

$$y_h = A e^{\int p(x)dx}$$

propose $y = A(x)\, e^{\int p(x)dx}$

substitute in the equation $y' + py = q$

get $A' = q\, e^{\int p dx}$

Nonlinear first order ODE

$$F(x, y, y') = 0$$

$\Downarrow$ assume you can express this as

$$y' = f(x, y)$$

Separable equations

$$y' = X(x)\,Y(y)$$

$$\Rightarrow \frac{dy}{Y(y)} = \frac{dx}{X(x)}$$

Exact equations

$$\boxed{y' = -\frac{M(x,y)}{N(x,y)}}$$

$$\Downarrow$$

$$M\,dx + N\,dy = 0$$

$\equiv$? Exact differential,

$$dF = 0$$

$$\boxed{F = C}$$

Test for exactness : $Mdx + Ndy$ is an exact differential iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof: $\cancel{\Leftarrow}$ $Mdx + Ndy$ exact $\Rightarrow$ F exists.

$$\Rightarrow \quad M = \frac{\partial F}{\partial x} \quad N = \frac{\partial F}{\partial y} \quad \Rightarrow M_y = N_x = F_{xy}$$

(everybody has been assumed continuous)

Use section 16.10 to prove that $M_y = N_x \Rightarrow$ $Mdx + Ndy$ is exact.

Integrating factor.

Suppose $M_y \neq N_x \Rightarrow$

Find $\sigma \ni$ $\begin{cases} \sigma(Mdx + Ndy) = 0 \\ \dfrac{\partial(\sigma M)}{\partial y} = \dfrac{\partial(\sigma N)}{\partial x} \end{cases}$

let $\sigma = \sigma(x)$

$$\Rightarrow \sigma M_y = \sigma_x N + \sigma N_x$$

$$\frac{d\sigma}{dx} = \sigma\left(\frac{M_y - N_x}{N}\right)$$

Then, needs to be $f(x)$ only.

$\sigma = \sigma(y)$

$$\sigma_y M + \sigma M_y = \sigma N_x$$

$$\frac{d\sigma}{dy} = -\sigma\left(\frac{M_y - N_x}{M}\right)$$ ← Then needs to be $f(y)$ alone.

Second order ODE

Linear Dependence:

$\{u_1, \ldots u_n\}$ are l.i. ~~iff~~ $\boxed{\alpha_i u_i(x) = 0}$

check the proof in Greenberg.

Practical ways of determining l.i.

Solve $\alpha_i u_i(x_j) = 0$ for $n \neq x_j$ points.

or. solve $\alpha_i u_i^{(j)}(x) = 0$ for $j = 1, \ldots (n-1)$

$$\Rightarrow W[u_1 \ldots u_n](x) = \begin{vmatrix} u_1 & \cdots & u_n \\ u_1' & \cdots & u_n' \\ \vdots & & \vdots \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$$

↑ Wronskian

If $W \neq 0$ for some $x = x_0 \Rightarrow \alpha_i = 0 \ \forall i$

let

$$Ly = y^{(n)} + p_1 y^{(n-1)} + \cdots p_n y = 0$$

IVP $\begin{cases} y(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_n \end{cases}$

BVP contain conditions on other points.

$\Rightarrow$ It has exactly n l.i. solutions

$\Rightarrow$ general solution is

$$y = c_i y_i$$

SOLUTIONS

Homogeneous Equation

p_i = constant.

$$\boxed{y'' + a_1 y' + a_2 y = 0}$$

Try $y = e^{\lambda x}$

$\Rightarrow \lambda^2 e^{\lambda x} + \lambda a_1 e^{\lambda x} + a_2 e^{\lambda x} = 0$

$\Rightarrow \lambda^2 + \lambda a_1 + a_2 = 0 \quad \begin{matrix} \nearrow \lambda_1 \\ \searrow \lambda_2 \end{matrix}$

$\lambda_1 \neq \lambda_2 \rightarrow y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$\lambda_1 = \lambda_2 \rightarrow y(x) = C_1 e^{\lambda x} + ?$

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Use reduction of order.
(Variation of parameter)

$$y_2 = C_1(x)\, e^{\lambda x}.$$

$$y_2' = C_1' e^{\lambda x} + \lambda C_1 e^{\lambda x}$$

$$y_2'' = C_1'' e^{\lambda x} + 2\lambda C_1' e^{\lambda x} + \lambda^2 C_1 e^{\lambda x}$$

$$y_2'' + a_1 y_2' + a_2 y_2 = (C_1'' + C_1' a_1 + C_1 a_2)\, e^{\lambda x} + (2\lambda C_1' + \lambda^2 C_1 + a_1 \lambda C_1)\, e^{\lambda x} = 0$$

$$C_1'' + C_1' \cdot (a_1 + 2\lambda) + C_1 \cancel{(a_2 + a_1\lambda + \lambda^2)} = 0$$

$$\Rightarrow C_1'' + C_1' (a_1 + 2\lambda) = 0$$

but $\lambda = -a_1/2$

$$\Rightarrow C_1'' = 0 \Rightarrow \boxed{C_1 = A + Bx}$$

In general

if λ is a root of order k.

$\Rightarrow$ $xe^{\lambda x}, x^2 e^{\lambda x}, x^3 e^{\lambda x} \dots x^{k-1} e^{\lambda x}$

are solutions.

Prove it

Cauchy-Euler equation

$$x^n \frac{d^n y}{dx^n} + c_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots c_n y = 0$$

If one change variables to $\xi = \ln x$

$$\Rightarrow \frac{d^n y}{d\xi^n} + c_1 \frac{d^{n-1} y}{d\xi^{n-1}} + \quad \dots \quad + c_n y = 0$$

is linear.

$\Rightarrow$ Fundamental solution is

$$e^{\lambda \ln x} = x^\lambda$$

Non homogeneous solution to $Ly = f$

$$L \equiv \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2$$

Undetermined coefficients

Make a linear combination of the successive derivatives of f.

eg. For example, if $f = xe^{-2x} \rightarrow f' = -2xe^{-2x} + e^{-2x}$
$f'' = 4xe^{-2x} + \ldots$

Propose a solution containing

Variation of Parameters

$$y_h = \sum c_i y_i$$
$$y_p = \sum c_i(x) y_i$$

For order 2.

Make this zero.

$$y_p' = c_1 y_1' + c_2 y_2' + c_1' y_1 + c_2' y_2$$

$$y_p'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'$$

$$\Rightarrow \boxed{c_1 \underbrace{Ly_1}_{=0} + c_2 \underbrace{Ly_2}_{=0} + c_1' y_1' + c_2' y_2' = f}$$

$$\Rightarrow \boxed{\begin{aligned} y_1 c_1' + y_2 c_2' &= 0 \\ y_1' c_1' + y_2' c_2' &= 0 \end{aligned}}$$

system of linear equations

$$c_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_1(x)}{W(x)} \qquad c_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_2' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_2(x)}{W(x)}$$

$$\Rightarrow y_p(x) = \left[\int \frac{W_1(\xi)}{W(\xi)}\, d\xi\right] y_1(x) + \left[\int \frac{W_2(\xi)}{W(\xi)}\, d\xi\right] y_2(x)$$

Can we extend this to higher order?

Systems of ODE

Let $L = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_n(t)$

New notation

$$L \equiv a_0(t) D^n + a_1(t) D^{n-1} + \cdots + a_n(t)$$

when $a_0(t) \ldots a_n(t)$ are constants one can have monomial expansions

$$(2D^2 + 5D - 3) = (2D - 1)(D + 3)$$

Also

$$L_1 L_2 = L_2 L_1$$

Thus we can use <u>elimination</u>

$$x' - x - y = 3t$$
$$x' + y' - 5x - 2y = 5$$

$$\Downarrow$$

$$(D-1)\,x - y = 3t$$
$$(D-5)\,x + (D-2)\,y = 5$$

$$\Downarrow$$

$$L_1 x + L_2 y = 3t$$
$$L_3 x + L_4 y = 5$$

$$\Downarrow$$

$$L_3(L_1 x + L_2 y) = L_3(3t)$$
$$L_1(L_3 x + L_4 y) = L_1(5)$$

$$\Rightarrow (L_1 L_4 - L_3 L_2)\,y = L_1(5) - L_3(3t)$$

$\Downarrow$ You can continue from here!

But can use Cramer's rule

$$x = \frac{\begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix}}{\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}}{\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix}}$$