Non homogeneous Sturm Liouville Problems

Let $Ly=0$ be a sturm Liouville homogeneous problem.

We know the solutions are orthogonal. Moreover they can be used to expand any function $f(x)$.

$$f(x) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n(x)$$

$\Rightarrow$ Let $Ly = f(x)$

$$\Rightarrow \int_a^b (\underbrace{Ly - f}_{=0}) \phi_n(x)\,dx = 0$$

$$\Rightarrow \int_a^b (p(x)y')'\phi_n(x)\,dx + \int_a^b q(x)\,y\,\phi_n(x)\,dx$$

$$+ \int_a^b \lambda w(x) y(x)\,dx - \int_a^b f\,\phi_n(x)\,dx = 0$$

But

$$\int_a^b (py')'\phi_n(x)\,dx = py'\phi_n\Big|_a^b - \int_a^b py'\phi_n'\,dx$$

$$= py'\phi_n\Big|_a^b - yp\phi_n'\Big|_a^b + \underbrace{\int_a^b (\phi_n' p)'y\,dx}_{\Downarrow}$$

$$= \int_a^b [-q(x)y\phi_n(x) - \lambda_n w(x)\phi_n(x)]\,dx$$

$$\Rightarrow \int_a^b (py')'\phi_n\,dx + \int_a^b q\,y\phi_n\,dx + \int_a^b \lambda w y\,dx$$

$$- \int f\phi_n\,dx = py'\phi_n\Big|_a^b - yp\phi_n'\Big|_a^b$$

$$- \cancel{\int_a^b q y\phi_n\,dx} - \int_a^b \lambda_n w\phi_n\,dx + \cancel{\int_a^b q y\phi_n\,dx}$$

$$+ \int_a^b \lambda w y\,dx - \int f\phi_n\,dx = 0$$

$$\Rightarrow \underbrace{py'\phi_n\Big|_a^b - yp\phi_n'\Big|_a^b}_{\alpha} + (\lambda - \lambda_n)\int_a^b w\phi_n\,dx = \int_a^b f\phi_n\,dx$$

$$\alpha + (\lambda - \lambda_n) \int_a^b w \phi_n y \, dx = \int_a^b f \phi_n \, dx$$

However, if

$$y = \sum_1^\infty c_n \phi_n(x) \qquad f(x) = \sum b_n \phi_n(x)$$

$$\Rightarrow \int_a^b w \phi_n y \, dx = c_n.$$

$$\Rightarrow \int_a^b f \phi_n \, dx = b_n.$$

$$\Rightarrow \boxed{b_n = (\lambda - \lambda_n) c_n + \alpha}$$

$$\Rightarrow \quad c_n = \frac{b_n}{\lambda - \lambda_n} - \frac{\alpha}{\lambda - \lambda_n}$$

$$\Rightarrow y(x) = \sum_{n=1}^\infty \frac{1}{\lambda - \lambda_n} (b_n - \alpha)$$

α can now be obtained from boundary conditions