Matrices: Eigenvalues, Eigenvectors

1) λ is an eigenvalue of $\underline{\underline{A}}$ iff $(A-\lambda I)$ is singular $\Rightarrow \det(A-\lambda I)=0$.

2) $\underline{p_i}$ is an eigenvector of A associated to λ_i iff $(A-\lambda_i I)\, p_i = 0$

* $\det(A-\lambda I)$ is the characteristic polynomial

Example $A = \begin{bmatrix} 1 & 1 & 0 \\ \bullet & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ \bullet & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (1-\lambda)(2-\lambda)(3-\lambda) \cancel{-}\,-(3-\lambda) = 0$$

(2)

Another example

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3$$

$$= \lambda^2 - \underset{\substack{\uparrow \\ tr\,A}}{6}\lambda + \underset{\substack{\uparrow \\ |A|}}{5} = 0 \quad \Rightarrow \lambda = \{$$

<u>Eigenvectors</u>

$\lambda_1 = 1$

$$(A - \lambda_1 I) p_1 = \begin{pmatrix} 2-\lambda_1 & 3 \\ 1 & 4-\lambda_1 \end{pmatrix} p_1 =$$

$$= \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} p_1 = 0$$

$$= \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = 0$$

$$\Rightarrow \boxed{p_{11} + 3 p_{12} = 0}$$

$$\Rightarrow \left. \begin{array}{l} p_{11} = -3 \\ p_{12} = 1 \end{array} \right\} \quad p_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

A 2×2 matrix need not have 2 eigenvalues.

Example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$|A - \lambda I| = \lambda^2 \Rightarrow \lambda = 0$$

Eigenvector $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

SIMILARITY

A, B (matrices) are similar ($A \overset{s}{\sim} B$) $\Leftrightarrow$ exist a nonsingular $P \ni$

$$B = P^{-1} A P.$$

Proposition: Let $A \overset{s}{\sim} B$. Then the families of eigenvalues are the same.

Proof: Since $A \overset{s}{\sim} B \Rightarrow B = P^{-1} A P$ for some P nonsingular

Then $(B - \lambda I) = P^{-1} (A - \lambda I) P$

Then $|B - \lambda I| = |P^{-1}(A - \lambda I) P| = |P^{-1}||A - \lambda I||P|$
$= |A - \lambda I|$

QED

<u>Proposition</u> $A \overset{s}{\sim} B \Rightarrow y$ is an eigenvector of B belonging to λ iff Py is an eigenvector of A belonging to λ.

<u>Proof.</u> $B = P^{-1}AP \Rightarrow$

$$\Rightarrow By = P^{-1}APy \quad \text{But } By = \lambda y$$

$$\Rightarrow P^{-1}APy = \lambda y \Rightarrow A\underbrace{Py} = \lambda \underbrace{Py}$$

definition of eigenvector

QED

$$\boxed{\text{Similar matrices have the same family of eigenvalues and corresponding eigenvectors}}$$

<u>Theorem</u>: Let A be a matrix with complex coefficients. If $P^{-1}AP = \Lambda$ where $\Lambda = \mathrm{diag}(\lambda_1 \cdots \lambda_n) \Rightarrow (\lambda_1 \cdots \lambda_n)$ is a family of eigenvalues of A.

<u>Proof</u>: use proposition above!

We say A is diagonizable if there exists $P \ni P^{-1}AP = \Lambda$ (Λ diagonal)

Repeated ~~ex~~ roots of $|A - \lambda I| = 0$

Let $|A - \lambda I| = (\lambda_1 - \lambda)^{\alpha_1} \cdots (\lambda_m - \lambda)^{\alpha_m} = 0$

$\Rightarrow$ α_i is the algebraic multiplicity of λ_i.

In addition, let $\gamma_i(\lambda_i)$ be the number of eigenvectors of A corresponding to λ_i.

We call $\gamma_i(\lambda_i)$ the geometric multiplicity of λ_i.

Lemma: Let $\lambda_1 \cdots \lambda_s$ be distinct eigenvalues of a matrix A, let $x^i, i=1..s$ be the eigenvectors ~~of~~ corresponding to λ^i $\Rightarrow (x^1 \cdots x^s)$ are linearly independent

Proof: Induction. 1) True for $s=1$ ($x^1 \neq 0$)

2) We need to prove that if true for $s \geq 2$, then it is true for $S = S+1$

True for s, ~~Then~~ $0 = \alpha_1 x^1 + \cdots \alpha_{s-1} x^{s-1}$ (6)

$\Rightarrow \alpha_1 = \alpha_2 \cdots = \alpha_{s-1} = 0$

Now, let

$\alpha_1 x^1 + \cdots \alpha_s x^s = 0$ (*)

we will prove $\alpha_1 = \alpha_2 = \cdots \alpha_s = 0$

$\alpha_1 x^1 \cdots + \alpha_s x^s = 0 \Rightarrow$

$\Rightarrow A(\alpha_1 x^1 + \cdots \alpha_s x^s) =$

$= \alpha_1 \lambda_1 x^1 + \qquad \alpha_s \lambda_s x^s = 0$

Multiply $(\alpha_1 x^1 \qquad + \alpha_s x^s) = 0$ by λ_s and subtract from ...

$\Rightarrow \alpha_1(\lambda_1 - \lambda_s) x^1 + \cdots \alpha_{s-1}(\lambda_{s-1} - \lambda_s) x^{s-1} = 0$

However these are l. i. $\Rightarrow$

$\alpha_i(\lambda_i - \lambda_s) = 0$, but $\lambda_i \neq \lambda_s \Rightarrow \alpha_i = 0$

$\forall i = 1, \ldots (s-1)$, which reduces (*) to

$\alpha_s x^s = 0 \Rightarrow \alpha_s = 0$. QED

Theorem: Let $\lambda_1 \ldots \lambda_s$ distinct eigenvalues of A, and let $x^{i1} \ldots x^{i r_i}$ the eigenvectors of λ_i

$\Rightarrow \quad (x^{11} \ldots x^{1 r_1} \ldots x^{s1} \ldots x^{s r_s})$

are l.i.

Proof:

Suppose $\sum_{i=1}^{s} \sum_{j=1}^{p_i} \alpha_{ij} x^{ij} = 0$

Need to prove $\alpha_{ij} = 0 \quad \forall ij$.

Let $x^i = \sum_{j=1}^{p_i} \alpha_{ij} x^{ij}$. Now, x^i is an eigenvector belonging to λ_i, or $x^i = 0$.

Suppose $x^i \neq 0$ for some $i = 1, \ldots r$.

$\Rightarrow \sum_{i=1}^{r} x^i = 0$, which is impossible by the previous Theorem. $\Rightarrow x^i = 0 \ \forall i$

$\Rightarrow \alpha_{ij} = 0$

QED

<u>Theorem</u> A is diagonable iff $\alpha_i = \gamma_i$

<u>Proof:</u> (a loosely speaking one).

a) Let $\alpha_i = \gamma_i \Rightarrow n = \sum \alpha_i = \sum \gamma_i$

but $(x^{i1} \ldots x^{i\gamma_i})$ are l.i.

$$\Rightarrow (x^{11} \ldots x^{1\gamma_1}, \ldots x^{m1} \ldots x^{m\gamma_m})$$

are l.i. (previous theorem). $\Rightarrow$ The matrix $P = [x^{11} \quad \ldots \quad x^{m\gamma_m}]$ will render the desired result.

b) A is diagonable. $\Rightarrow$ P exists.

$\Rightarrow$ There are n vectors qualifying to be eigenvectors.

JORDAN FORMS

Define $U_r = [u_{ij}] \ni u_{i,i+1} = 1$
$u_{ij} = 0$ otherwise

$$U_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad U_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

etc. first superdiagonals

<u>Jordan Block</u>: A matrix of the form $\lambda I_r + U_r$

<u>Definition</u>: a direct sum of two matrices is

$$A \oplus B = \left[\begin{array}{c|c} A & 0 \\ 0 & B \end{array}\right]$$

<u>Definition</u> A matrix is in a Jordan canonical form if it is a direct sum of Jordan blocks

$$\Lambda = (\lambda_1 I_{r_1} + U_{r_1}) \oplus \cdots \oplus (\lambda_k I_{r_k} + U_{r_k})$$

Example

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (0 I_2 + U_2) \oplus (2 I_1 + U_1)$$

<u>Theorem</u>: Let A be a matrix. A matrix Λ in Jordan canonical form exists such that $\Lambda \overset{s}{\sim} A$.

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We have some candidates for columns of P. They are the eigenvectors. What happens when Λ is a Jordan form. Need to find other columns of P.

know A,
know Λ

$$P = \begin{bmatrix} p_1 & \underbrace{\cdots}_{} & p_2 & \cdots \end{bmatrix}$$

missing

$s = \#$ of $\neq$ eigenvectors

$\Rightarrow n-s$ vectors to be determined.

$\Rightarrow (n-s) \times n$ unknowns

but $A = P^{-1} \Lambda P$
$PA = \Lambda P \Rightarrow$ <u>n^2 equations</u>

Functions of Matrices

Let $f(\lambda) = \sum c_i \lambda^i$

we *define* $f(A) = \sum c_i A^i$ $\quad \boxed{A^0 \triangleq I}$

Suppose $B = P^{-1} A P$

$\Rightarrow B^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$

$\Rightarrow B^r = P^{-1}A^rP$

What do I use here?

$$\Rightarrow f(B) = \sum c_i (P^{-1}AP)^i = $$
$$= \sum c_i P^{-1}A^iP = P^{-1}\sum c_i A^i P$$
$$= P^{-1} f(A) P$$

It also works if f is an ∞ series

Hamilton Cayley Theorem

Every matrix satisfies its own characteristic equation.

Proof: We will prove it for A diagonable only.

Let $|A - \lambda I| = p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$

$\Rightarrow A = P^{-1} \Lambda P \Rightarrow p(A) = P^{-1} p(\Lambda) P$

but

$$P(\Lambda) = (\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I) \cdots (\Lambda - \lambda_n I)$$

However

$$(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I) = \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & \lambda_3^2 & \\ 0 & 0 & & \ddots \ \lambda_n^2 \end{bmatrix}$$

In other words $P(\Lambda) = 0$

$$\Rightarrow P(A) = 0$$

QED

It is also true for A nondiagonable.

<u>Consequence</u>:

let $f(\lambda)$ be any polynomial

$$f(\lambda) = g(\lambda) + h(\lambda) P(\lambda)$$

$$\Rightarrow f(A) = g(A) + h(A) \underbrace{P(A)}_{=0}$$

example

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -3 & 3 & 1 \\ -4 & 3 & 2 \end{bmatrix} \qquad p(\lambda) = (1-\lambda)^2 (2-\lambda)$$

let $f(\lambda) = (1-\lambda)^4 (2-\lambda)^3 + (\lambda+1)$

$$\Rightarrow f(t) = (2-\lambda)p(\lambda)^2 + (\lambda+1)$$

$$\Rightarrow f(A) = (I-A)^4 (2I-A)^3 + (A+I) =$$

$$= g(A) = A + I$$

— o —

$A^r = ?$ $\quad p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n$ (13)

$$\Rightarrow \left| A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \dots - a_n I \right|$$

which can be used recursively

However, note that if λ_i is an eigenvalue of $A \Rightarrow \lambda^m$ is an eigenvalue of A^m

$$\Rightarrow p(\lambda^m) = \lambda^{nm} + a_1 \lambda^{m(n-1)} \dots + a_n$$

$$\Rightarrow A^r = -a_1 A^{r-m} - a_2 A^{r-2m} - \dots - a_n I$$

Also,

$$I = -\frac{1}{a_n}\left[A^n + a_1 A^{n-1} \cdots + a_{n-1} A\right]$$

$$\Rightarrow \underbrace{A^{-1} = -\frac{1}{a_n}\left[A^{n-1} + a_1 A^{n-2} \cdots + a_{n-1} I\right]}_{\text{Remarkable!}}$$

but very seldom used

— o —

Finally, we said $f(B) = P^{-1} f(A) P$ is valid for series.

$$\Rightarrow e^{A} = I + A + \frac{A^2}{2!} + \cdots \frac{A^n}{n!} + \cdots$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \cdots \frac{A^n t^n}{n!}$$

Moreover

$$\frac{d}{dt}\left[e^{At}\right] = A e^{At}$$

Immediate consequence for systems of ODE's.

$x' = Ax$ let $A = P^{-1} \Lambda P$

$\Rightarrow x' = P^{-1} \Lambda \underbrace{Px}_{=y} = P^{-1} \Lambda y$

$\Rightarrow y' = Px' \Rightarrow y' = \Lambda y$

diagonal

$\Rightarrow y_i' = \lambda_i y_i \Rightarrow y_i = c_i e^{\lambda_i}$

$\Rightarrow y = e^{\Lambda} c_1$ but $y = Px$

vector of constants

$\Rightarrow x = P^{-1} y = P^{-1} e^{\Lambda t} c_1$

However $e^{At} = P^{-1} e^{\Lambda t} P \Rightarrow$ This is what we call elegance!

$P^{-1} e^{\Lambda t} = e^{At} P^{-1}$

$\Rightarrow \boxed{x = e^{At} P^{-1} c_1 = e^{At} c}$

Let us apply variation of parameters to solve

$$x' = Ax + g$$

Assume Φ is a fundamental matrix of the ODE system (A constant $\Rightarrow \Phi = e^{At}$)

Let $\varphi_p = \phi h$.

$$\varphi_p' = A\phi h + g.$$

$$\Downarrow$$

$$\phi' h + \phi h' = A\phi h + g$$

$$\Downarrow$$

$$\cancel{A\phi h} + \phi h' = \cancel{A\phi h} + g \Rightarrow \phi h' = g; \quad h' = \phi^{-1} g$$

$$\Rightarrow h = \int \phi^{-1}_{(s)} g(s)\, ds$$

$\Rightarrow$ General solution is

$$\boxed{\tilde{\phi} = \phi \underline{c} + \phi \int_{t_0}^{t} \phi^{-1} g \, ds}$$

obtained from initial conditions

Conditions at two points

$$\begin{cases} \dot{x} = Ax + g \\ W_a x(a) + W_b x(b) = b \end{cases}$$

Let $D = W_a \phi(a) + W_b \phi(b)$

fundamental matrix

but

$$\tilde{\phi} = \phi c + \underbrace{\phi \int_a^t \phi^{-1} g \, ds}_{\varphi} = \phi c + \varphi$$

$$\Rightarrow W_a \tilde{\phi}(a) + W_b \tilde{\phi}(b) = Dc + e = b$$

where $e = W_a \overbrace{\cancel{\varphi(a)}}^{=0} + W_b \varphi(b)$

$\Rightarrow$ Need to solve

$$\boxed{Dc = b - e}$$

D needs to be of full rank.

Symmetric Matrices

$A^T = A$

1) All eigenvalues are real
2) λ is of multiplicity k $\Rightarrow$ eigenspace is of dimension k
3) eigenvectors are orthogonal.

Proof

1) $Ae = \lambda e$ $\Rightarrow (Ae)^T \bar{e} = (\lambda e)^T \bar{e} = \lambda e^T \bar{e}$

$A\bar{e} = \bar{\lambda}\bar{e}$ $\Rightarrow (A\bar{e})^T e = (\bar{\lambda}\bar{e})^T e = \bar{\lambda}\bar{e}^T e$

$$0 = (\lambda - \bar{\lambda}) e^T \bar{e}$$

$\hookrightarrow \lambda = \bar{\lambda}$

$\Rightarrow \lambda$ is real

2) Proved previously (they are l.i.)

3) $Ae_j = \lambda_j e_j$ $\qquad A e_k = \lambda_k e_k$

$\Downarrow \qquad\qquad \Downarrow$

$e_k^T A e_j = e_k^T \lambda_j e_j \qquad (Ae_k)^T = \lambda_k e_k^T$

$\Downarrow$

$e_k^T A^T e_j = \lambda_k e_k^T e_j$

$$\Rightarrow e_k^T A e_j = e_k^T \lambda_j e_j$$
$$e_k^T A^T e_j = e_k^T \lambda_k e_j$$

$A = A^T \Rightarrow 0 = (\lambda_j - \lambda_k) e_k^T e_j \Rightarrow e_k^T e_j = 0$

QED

Hermitian Matrices

$$A^{*} \equiv \bar{A}^{T}$$

hermitian conjugate of A

If $\bar{A} = A \Rightarrow A$ is real

If $A^{*} = A \Rightarrow A$ is hermitian

1) Hermitian matrices have real eigenvalues

2) " " " orthogonal eigenvectors