Linear PDE (2nd Order)

General equation

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = f.$$

- Parabolic $B^2 - AC = 0$

  e.g. Diffusion type equation

  $$\boxed{\alpha^2 u_{xx} = u_t}$$

- Hyperbolic. $B^2 - AC > 0$

  $c^2 u_{xx} = u_{tt}$ ← wave equation

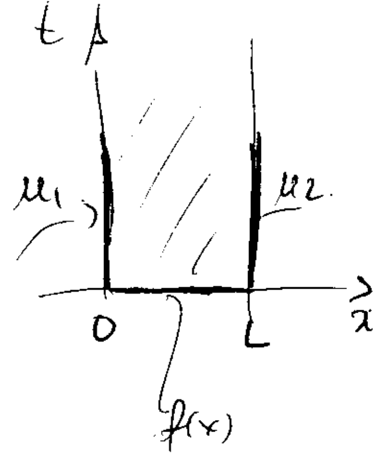
- Elliptic $B^2 - AC < 0$

  $u_{xx} + u_{yy} = 0$ ← heat equation (Laplace equation)

Separation of variables

Diffusion Problem

$$\alpha^2 u_{xx} - u_t = 0$$
$$u(0,t) = u_1$$
$$u(L,t) = u_2$$
$$u(x,0) = f(x)$$



Let $u = X(x)\,T(t)$

$$\Rightarrow \alpha^2 X'' T = X T'$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}$$

$$F(x) = G(t) \quad \Rightarrow F(x) = G(t) = -\lambda^2$$

$$\Rightarrow \begin{cases} X'' + \lambda^2 X = 0 \\ T' + \lambda^2 \alpha^2 T = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X = A\cos\lambda x + B\sin\lambda x \\ T = C e^{-\lambda^2\alpha^2 t} \end{cases} \qquad \lambda \neq 0$$

$$\left.\begin{array}{l} X = D + Ex \\ T = G \end{array}\right\} \lambda = 0$$

$$\Rightarrow \begin{cases} u = H + Ix & \lambda = 0 \\ u = (J\cos\lambda x + K\sin\lambda x)\, e^{-\lambda^2\alpha^2 t} \end{cases}$$

By the superposition principle

$$u = (H + Ix) + \sum_{n=1}^{\infty} (J_n \cos\lambda_n x + K_n \sin\lambda_n x)\, e^{-\lambda_n^2\alpha^2 t}$$

Use BC

$$u(0,t) = u_1 \Rightarrow u_1 = H + \sum_{n=1}^{\infty} J_n e^{-\lambda_n^2 \alpha^2 t}$$

$$\Rightarrow (H - u_1) + \sum_{n=1}^{\infty} J_n e^{-\lambda_n^2 \alpha^2 t} = 0$$

~~But (1)~~

But (1) and $e^{-\lambda_n^2 \alpha t}$ are l.i.

$$\Rightarrow \left.\begin{array}{l} H - u_1 = 0 \\ J_n = 0 \end{array}\right\} \Rightarrow$$

$$u = (u_1 + Ix) + \sum_{n=1}^{\infty} K_n \sin\lambda_n x$$

$u(L,t) = u_2 \Rightarrow$

$$u_2 = (u_1 + I\cancel{L}) + \sum_{n=1}^{\infty} K_n \sin(\lambda_n L) \cdot e^{-\lambda_n^2 \alpha^2 t}$$

$$(u_1 + IL - u_2)(1) + \sum_{l.i} K_n \sin(\lambda_n L)\, e^{-\lambda_n^2 \alpha^2 t}$$

$\Rightarrow$

$$u_1 + IL - u_2 = 0 \Rightarrow I = \frac{u_2 - u_1}{L}$$

$$K_n \sin \lambda_n L \cancel{=} 0 \Rightarrow \begin{cases} K_n = 0 \\ \text{or} \\ \sin \lambda_n L = 0 \end{cases}$$

$K_n = 0 \Rightarrow$

$$\boxed{u = u_1 + \frac{u_2 - u_1}{L} x} \Rightarrow$$

but $u(x,0) = f(x) \Rightarrow$ if $f(x) = u_1 + \frac{u_2 - u_1}{L} x$

Then this is acceptable.

$\Rightarrow$ choose in general

$$\sin \lambda_n L = 0 \Rightarrow \lambda_n = \frac{n\pi}{L}$$

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$$u(x,t) = u_1 + (u_2 - u_1)\frac{x}{L} + \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}$$

Thus

$$u(x,0) = f(x) = u_1 + (u_2 - u_1)\frac{x}{L} + \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L}$$

$$K_n = ?$$

Let $\underbrace{\left\{ f(x) - u_1 - (u_2 - u_1)\frac{x}{L} \right\}}_{F(x)} = \underbrace{\sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L}}_{\text{Fourier expansion of } F(x)?}$

$$\Rightarrow K_n = \frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}\,dx$$

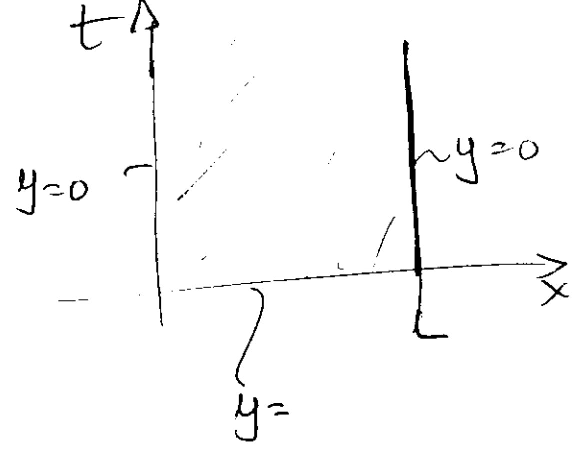
Vibrating String

$c^2 y_{xx} = y_{tt}$

$y(0,t)=0$

$y(L,t)=0$

$y(x,0)=f(x)$



$$y(x,t) = X(x)\,T(t)$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

– Rest of story is the same.

Laplace Equation

$\nabla^2 u = 0$

$\downarrow$

$u_{xx} + u_{yy} = 0$

$u(0,y) = 0$

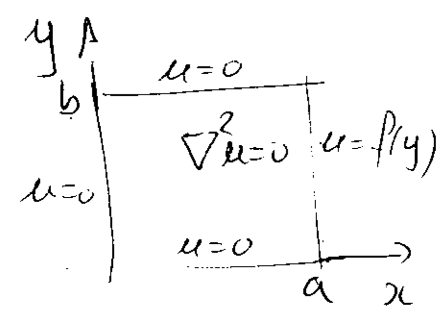
$u(a,y) = f(y)$

$u(x,0) = u(x,b) = 0$

Same story –

Poisson Equation

$\nabla^2 u = f$



Presence of source term

$\alpha^2 u_{xx} = u_t - g(x,t)$

Let $u = \sum_{n=0}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$ ← same form as homogeneous solution

$g(x,t) = \sum_{n=0} g_n(t) \sin \frac{n\pi x}{L}$

$$g_n(t) = \frac{2}{L} \int_0^L g(x,t) \sin \frac{n\pi x}{L} dx$$

$\Rightarrow$

$$-\alpha^2 \sum_{n=0}^{\infty} a_n(t) \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} = \sum_{n=0}^{\infty} a_n'(t) \sin \frac{n\pi x}{L} - \sum_{n=0}^{\infty} g_n(t) \sin \frac{n\pi x}{L}$$

$$\Rightarrow + \left(\frac{n\pi}{L}\right)^2 \alpha^2 a_n(t) + a_n'(t) - g_n(t) = 0$$

$$\Rightarrow \boxed{a_n'(t) + \left(\frac{n\pi}{L}\right)^2 \alpha^2 a_n(t) = g_n(t)}$$

⇑ integrate and you are done!

<u>Other cases. same Thing!</u>

General equation

$$A\,u_{xx} + 2\,B u_{xy} + C\, u_{yy} + D u_x + E u_y + F u = R$$

problem?

know how to deal with this as long as have $u = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$

orthogonal

Indeed

$$A u_{xx} + C u_{yy} + D u_x + E u_y + F u = 0$$

$$u = XY \Rightarrow A\frac{X''}{X} + C\frac{Y''}{Y} + D\frac{X'}{X} + E\frac{Y'}{Y} + F = 0$$

$$\Rightarrow A\frac{X''}{X} + D\frac{X'}{X} + F = -C\frac{Y''}{Y} - E\frac{Y'}{Y}$$

here or here

This means we can use cylindrical or spherical coordinates!

$\Rightarrow$ works as long as

$A = A(x)$ $\quad C = C(y)$

$D = D(x)$ $\quad E = E(y)$

or A, C, D, E separable!

e.g. $A = A_1(x)A_2(y)$

u_{xy} term?

Change coordinates $z = x + \alpha y$
$s = x + \beta y$

$u_x = u_z \cancel{z_x} + u_s s_x = u_z + u_s$
$u_{xx} = (u_z + u_s)_{\cancel{z}} + (u_z + u_s)_s = u_{zz} + 2u_{zs} + u_{ss}$

$u_{xy} = (u_z + u_s)_z \alpha + (u_z + u_s)_s \beta =$
$= \alpha u_{zz} + (\alpha + \beta) u_{sz} + \beta u_{ss}$

$u_y = u_z \alpha + u_s \beta$
$u_{yy} = u_{zz}\alpha^2 + 2u_{s\cancel{z}}\beta\alpha + u_{ss}\beta^2$

$\Rightarrow A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u =$

$= A(u_{zz} + 2u_{zs} + u_{ss}) + 2B(\alpha u_{zz} + (\alpha+\beta)u_{sz} + \beta u_{ss})$
$+ C(u_{zz}\alpha^2 + 2\beta\alpha u_{sz} + u_{ss}\beta^2) +$
$+ \cdots = 0$

Need terms in u_{zs} dissapear

$\Rightarrow 2A + 2B(\alpha+\beta) + 2C\beta\alpha = 0$

let $\alpha = 1$

$$A + B(1+\beta) + C\beta = 0$$

$$\Rightarrow \beta = \frac{-(A+B)}{(C+B)}$$

$$\Rightarrow \boxed{A' u_{zz} + C' u_{ss} + \cdots = 0}$$

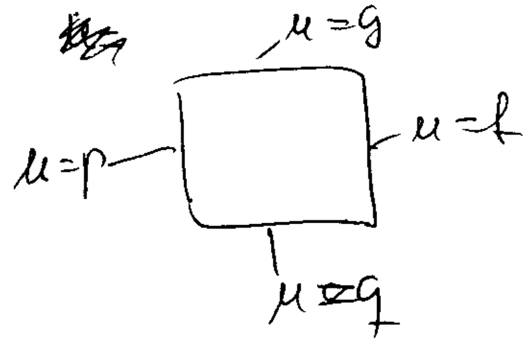
Only possible if A, B, C are constant!

<u>If not</u>: can we make? $z = \phi(x) + \psi(y)$
$s = \phi(x) + \Omega(y)$

or something similar?

Superposition

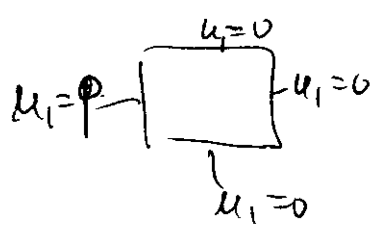
Let $\nabla^2 u = 0$

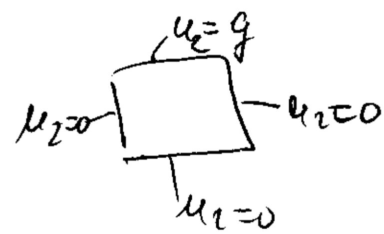


Since the equation is linear

Then $u = u_1 + u_2 + u_3 + u_4$

$\nabla^2 u_1 = 0$ $\qquad$ $\nabla^2 u_2 = 0$





and so on.

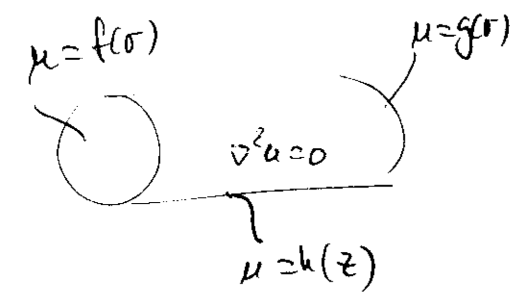
Cylindrical coordinates

$\nabla^2 u = 0$

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0$$

Suppose axisymmetry. $u_{\theta\theta} = 0$

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0$$



Use superposition

$$u = R Z$$

$$Z R_{rr} + \frac{1}{r} R_r Z + R Z_{zz} = 0$$

$$\frac{R'' + \frac{1}{r} R'}{R} = -\frac{Z''}{Z} = \text{constant} = k^2$$

The rest is the same.

Assume $u_{\theta\theta} \neq 0$

$$u = R Z \Theta$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} =$$

$$= R'' Z \Theta + \frac{1}{r} R' Z \Theta + \frac{1}{r^2} \Theta'' Z R + Z'' \Theta R = 0$$

$$\Rightarrow \frac{R'' + \frac{1}{r} R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0$$

$$\frac{R'' + \frac{1}{r} R'}{R} + \frac{1}{r} \frac{\Theta''}{\Theta} = - \frac{Z''}{Z} = -\lambda^2$$

$$\begin{cases} \dfrac{R'' + \frac{1}{r} R'}{R} + \dfrac{1}{r} \dfrac{\Theta''}{\Theta} = -\lambda^2 \\ Z'' - \lambda^2 Z = 0 \end{cases}$$

$$\frac{rR'' + R'}{R} + \lambda^2 r + \frac{\Theta''}{\Theta} = 0$$

$$\Rightarrow \quad r\frac{R''+R'}{R} + \lambda^2 r = -\frac{\Theta''}{\Theta} = -\mu^2$$

$$\Rightarrow \begin{cases} r\underline{R''+R'} + (\mu^2+\lambda^2 r)R = 0 \\ \Theta'' - \mu^2\Theta = 0 \end{cases}$$

In this problem you need to use all λ_n obtained for $z'' - \lambda^2 z = 0$