IRREGULAR SINGULAR POINTS

Consider $y'' + \frac{2}{x} y' + \frac{a}{x^4} y = 0 \qquad a > 0$

$x = 0$ is a singular point
Moreover it is an irregular singular point

The solution is
$$y = A \sin \frac{\sqrt{a}}{x} + B \cos \frac{\sqrt{a}}{x}$$

Note that there is no series expansion of y around $x = 0$.

Note $y = C e^{i\sqrt{a}/x}$

What if we did not know the solution.
If we had tried
$$y = e^{S(x)}$$

$$\Rightarrow y' = S'(x) e^{S} \Rightarrow y'' = e^{S} (S'(x))^2 + e^{S} S''(x) = y (S'(x))^2 + y S''(x)$$
$$y' = y S'$$

$$y\left[S'' + (S')^2 + \frac{2}{x}S' + \frac{a}{x^4}\right] = 0$$

we did not gain much

Assume further that

$$S = a\,x^{-b} \implies S' = -ba\,x^{-b-1}$$

$$\implies (S')^2 = b^2a^2\,x^{-2b-2}$$

$$S'' = +b(b+1)a\,x^{-b-2}$$

Note that

$$\frac{S''}{(S')^2} = \frac{\cancel{b}(b+1)\cancel{a}\,\cancel{x^{-(b+2)}}}{b^2a^2\,x^{-\cancel{(b+2)}-b}} = \frac{(b+1)}{ab}x^{b}$$

$$\implies \frac{S''}{(S')^2} \to 0 \quad \text{as } x \to 0$$

which is our irregular singular point.

WHY NOT then try ignoring S'' near $x=0$

$$(\hat{S}')^2 + \frac{2}{x}\hat{S}' + \frac{a}{x^4} \cong 0 \qquad \text{near } x=0$$

$$\implies \hat{S}' = -\frac{1}{x} \pm \sqrt{\frac{1}{x^2} - \frac{a}{x^4}} = -\frac{1}{x} \pm \frac{1}{x}\sqrt{1-\frac{a}{x^2}}$$

$$\hat{S}' = -\frac{1}{x} \pm \frac{1}{x^2}\sqrt{x^2 - a} \overset{\sim}{=} \pm \frac{i}{x^2}\sqrt{a}$$

$$\Rightarrow \hat{S} = \frac{\pm i\sqrt{a}}{x}$$ remarkable!

So the recipe seems to be

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ \text{Try } y = e^{S(x)} \\ \text{Ignore } S'' \ll (S')^2 \end{cases}$$

Example $x^3 y'' = y$.

$$\Rightarrow S'' + (S')^2 - \frac{1}{x^3} = 0 \qquad S'' \ll (S')^2$$

$$\Rightarrow \hat{S}' = x^{-3/2} \Rightarrow \hat{S} \doteq \pm 2x^{-1/2}$$

Choose + sign.

Can we improve the approximation?

Assume $S(x) = 2x^{-1/2} + C(x)$

presumable $C(x) \ll 2x^{-1/2}$ (near $x=0$)

Stick $y = e^{2x^{-1/2} + C(x)}$ into $x^3 y'' = y$

to obtain

$$\frac{3}{2} x^{-5/2} + C'' - 2x^{-3/2} C' + (C')^2 = 0$$

We want $C(x) \ll O(x^{-1/2})$

Then we need $C'(x) \ll O(x^{-3/2})$ $C'' \ll O(x^{-5/2})$

But $C' \ll O(x^{-3/2}) \Rightarrow (C')^2 \ll O(x^{-3/2})C'$

Therefore the dominant terms are

$$\frac{3}{2} x^{-5/2} - 2x^{-3/2} C' = 0$$

$$\Rightarrow C(x) = \frac{3}{4} \ln x$$

$$\Rightarrow y = e^{2x^{-1/2} + \frac{3}{4}\ln x} = x^{3/4} e^{2x^{-1/2}}$$

Can proceed again $y = e^{S(x) + C(x) + D(x)}$

and solve for $D(x)$

This route can be pursued, but there is a better one.

Assume $e^{D(x)}$ can be expanded in a form of a series

$$[1 + a x^{\alpha} + b x^{\beta} + \cdots]$$

$\Rightarrow$ Try $y = x^{3/4} e^{2x^{-1/2}} [1 + a_1 x^{\alpha}] = A(x)[1+a_1 x^{\alpha}]$

$$y' = A a_1 \alpha x^{\alpha-1} + A'[1+a_1 x^{\alpha}]$$

$$y'' = A a_1 \alpha(\alpha-1) x^{\alpha-2} + 2A' a_1 \alpha x^{\alpha-1} + A''[1+a_1 x^{\alpha}]$$

$$A' = \left(\frac{3}{4} x^{-1/4} - x^{+3/4} x^{-3/2}\right) e^{2x^{-1/2}} = \left(\frac{3}{4} x^{-1/4} - x^{-3/4}\right) e^{2x^{-1/2}}$$

$$A'' = \left(\frac{3}{4} x^{-1/4} - x^{-3/4}\right)\left(-x^{-3/2}\right) e^{2x^{-1/2}} + \left(-\frac{3}{16} x^{-5/4} + \frac{3}{4} x^{-7/4}\right) e^{2x^{-1/2}}$$

$$= \left[-\cancel{\frac{3}{4} x^{-7/4}} + x^{-9/4} - \frac{3}{16} x^{-5/4} + \cancel{\frac{3}{4} x^{-7/4}}\right] e^{2x^{-1/2}}$$

$$= \left[-\frac{3}{16} x^{-5/4} + x^{-9/4}\right] e^{2x^{-1/2}}$$

$$y''x^3 - y = e^{2x^{-1/2}} \Big\{ a_1\alpha(\alpha-1)\, x^{\alpha-2+\frac{3}{4}+3} + 2a_1\alpha\left[\tfrac{3}{4}x^{-1/4} - x^{-3/4}\right] x^{\alpha-1+3}$$

$$+ \left[-\tfrac{3}{16}x^{-5/4} + x^{-9/4}\right] x^3 \cdot (1 + a_1 x^\alpha) -$$

$$x^{3/4}\left[1 + a_1 x^\alpha\right] \Big\} \cong 0$$

neglect

$$\Rightarrow a_1\alpha(\alpha-1)\, x^{\alpha+7/4} + \tfrac{3}{2} a_1 \alpha x^{\alpha+7/4} - 2a_1\alpha\, x^{\alpha+5/4}$$

$$+ \left[-\tfrac{3}{16} x^{7/4} + x^{3/4}\right] (1 + \underbrace{a_1 x^\alpha}) +$$

$$+ x^{3/4}\left[1 + \underbrace{a_1 x^\alpha}\right] \cong 0$$

neglect

neglect

$$\Rightarrow \quad -2a_1\alpha\, x^{\alpha+5/4} + 2x^{3/4} \approx 0$$

$$\Rightarrow \alpha + \tfrac{5}{4} = \tfrac{3}{4} \quad \Rightarrow \boxed{\alpha = -\tfrac{1}{2}}$$

$$\Rightarrow -2a_1\alpha + 2 = 0$$

$$\Rightarrow \quad a_1 = \tfrac{1}{\alpha} \quad \Rightarrow \boxed{a_1 = -2}$$

Now can try $y = x^{3/4} e^{2x^{-1/2}} [1 - \frac{2}{x^{1/2}} + a_2 x^\beta] \ldots$

The recipe seems to work for $x = 0$

- stick $y = e^{S(x)} \Rightarrow$ get ODE in S
- use $S'' \ll (S')^2$

solve

- Propose $y = e^{S(x)} x^{\mu} \Rightarrow$ get ODE solve.

⋮

Behaviour of functions as $x \to \infty$

Consider
$$u'' + \frac{1}{x} u' + \left(1 - \frac{\nu^2}{x^2}\right) u = 0$$

∞ is an irregular singular point.

Why. consider $z = \frac{1}{x}$

$$\Rightarrow \frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = -\frac{1}{x^2}\frac{d}{dz} = -z^2 \frac{d}{dz}$$

$$\Rightarrow \frac{d^2}{dx^2} = \frac{d}{dz}\left(-z^2 \frac{d}{dz}\right)\frac{dz}{dx} = -z^2 \frac{d}{dz}\left(-z^2 \frac{d}{dz}\right) =$$

$$= z^4 \frac{d^2}{dz^2} + 2z^3 \frac{d}{dz}$$

$$\Rightarrow u'' + \frac{1}{x}u' + \left(1 - \frac{\nu^2}{x^2}\right)u = z^4 u_{zz} + 2z^3 u_z + z(-z^2) u_z + (1 - \nu^2 z^2) u = 0$$

$$\Rightarrow u_{zz} + \frac{1}{z} u_z + \underbrace{\left(\frac{1}{z^4} - \frac{\nu^2}{z^2}\right)}_{} u = 0$$

$z = 0$ is irregular singular!

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

$$y = e^{S(x)} \quad \text{as } x \to \infty$$

$$\Rightarrow S'' + (S')^2 + \frac{1}{x}S' + \left(1 - \frac{\nu^2}{x^2}\right) = 0$$

<u>Assume</u> $S'' \ll (S')^2$ $\quad S' = \frac{-1}{2x} \pm \sqrt{\frac{1}{4x^2} - 1}$

$$S' \to \pm i \quad \text{as } x \to \infty$$

$$\Rightarrow S(x) \sim \pm ix \quad \text{as } x \to \infty$$

Improve $\quad y = x^{\mu} e^{\pm ix}$ $\quad \left\{ \begin{array}{l} y' = (\mu x^{\mu-1} \pm i x^{\mu}) e^{\pm ix} \\ y'' = [\mu(\mu-1)x^{\mu-2} \pm i\mu x^{\mu}] e^{\pm ix} \\ \pm (\mu x^{\mu-1} \pm i x^{\mu}) i e^{\pm ix} \end{array} \right.$

$$\Rightarrow \Big\{ \mu(\mu-1)x^{-2} \pm 2i\mu x^{-1} \cancel{-1} + \mu x^{-2} \pm i x^{-1} + \cancel{1} - \nu^2 x^{-2} \Big\} x^{\mu} e^{\pm ix} \simeq 0$$

Now as $x \to \infty$ $\quad x^{-2} \ll x^{-1}$

$\Rightarrow$ Take leading terms ($O(x^{-1})$) and ignore the rest

$$\pm 2 i \mu x^{-1} \pm i x^{-1} = 0$$

$$\Rightarrow \mu = -1/2$$

$$\Rightarrow y \sim x^{-1/2} e^{\pm i x}$$

Can continue trying

$$y \sim x^{-1/2} e^{\pm i x} \left[1 + \frac{(\ )}{x^2} + \cdots \right]$$

Assumption $\overline{S''}^{\,0} \ll \overline{(S')^2}$ Does not work always!