Green Matrices and functions ①

Consider the problem

$$\begin{cases} \dot{x} = Ax + h \\ w_a\, x(a) + w_b\, x(b) = b \end{cases}$$

Example $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + ($

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(1) = ($

Recall we arrived at

$$\tilde{\phi} = \phi c + \phi \int_a^t \phi^{-1}(s)\, h(s)\, ds$$

$$\underbrace{D}_{w_a \phi(a) + w_b \phi(b)} c = b - \underbrace{e}_{w_b \phi(b) \int_a^b \phi^{-1}(s) h(s) ds} \implies c = D^{-1} b - D^{-1} e$$

$$\implies \tilde{\phi}(t) = \phi(t) D^{-1} b - \phi(t) D^{-1} w_b \phi(b) \int_a^b \phi^{-1}(s) h(s) ds$$
$$+ \phi(t) \int_a^t \phi^{-1}(s) h(s) ds$$

Rearranging.

$$\tilde{\phi}(t) = \phi(t) D^{-1} b + \int_a^t \phi(t) \phi^{-1}(s) h(s) ds - \int_a^t \phi(t) D^{-1} w_b \phi(b) \phi^{-1}(s) h(s$$
$$- \int_t^b \phi(t) D^{-1} w_b \phi(b) \phi^{-1}(s) h(s) ds$$

Now

$$\int_a^t \phi(t)\phi^{-1}(s)h(s)\,ds = I\int_a^t \phi(t)\phi^{-1}(s)h(s)\,ds$$

$$= D^{-1}\left(w_a\,\phi(a) + w_b\,\phi(b)\right)\int_a^t \phi(t)\phi^{-1}(s)h(s)\,ds$$

$$= \int_a^t D^{-1}w_a\,\phi(a)\,\phi(t)\,\phi^{-1}(s)\,h(s)\,ds \;+$$

$$+ \int_a^t D^{-1}w_b\,\phi(b)\,\phi(t)\,\phi^{-1}(s)\,h(s)\,ds$$

$$\Longrightarrow$$

$$\tilde{\phi} = \phi(t)D^{-1}b + \int_a^t \phi(t)D^{-1}w_a\,\phi(a)\,\phi^{-1}(s)h(s)\,ds$$

$$- \int_t^b \phi(t)\,D^{-1}w_b\,\phi(b)\,\phi^{-1}(s)\,h(s)\,ds$$

Let
$$G(t,s) = \begin{cases} \phi(t)\,D^{-1}w_a\,\phi(a)\,\phi^{-1}(s) & s \le t \\ -\phi(t)\,D^{-1}w_b\,\phi(b)\,\phi^{-1}(s) & s > t \end{cases}$$

Green Matrix

$$\Longrightarrow \boxed{\tilde{\phi}(t) = \int_a^b G(t,s)\,h(s)\,ds + \phi(t)D^{-1}b}$$

Back to example:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} t \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \dot{x}_1 = x_2 + t \\ \dot{x}_2 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1(0) = 1 \\ x_1(1) = 0 \end{cases}$$

Need ϕ, ϕ^{-1} solve $Ax = \dot{x}$

$\dot{x}_2 = 0 \Rightarrow x_2 = c_1 \Rightarrow \dot{x}_1 = x_2 = c_1$

$\Rightarrow x_1 = c_1 t + c_2$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 t + c_2 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} t \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \phi = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \phi^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -t \end{pmatrix}$$

Need D, D^{-1}

$$D = W_a \phi(a) + W_b \phi(b) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow D^{-1} b = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\int \phi^{-1}(s)\, h(s)\, ds = \int \begin{pmatrix} 0 & 1 \\ 1 & -s \end{pmatrix}\begin{pmatrix} s \\ 0 \end{pmatrix} ds = \int \begin{pmatrix} 0 \\ s \end{pmatrix} ds = \begin{pmatrix} 0 \\ s^2/2 \end{pmatrix}$$

$$\int_a^b \phi^{-1}(s)\, h(s)\, ds = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

$$\int_a^t \phi^{-1}(s)\, h(s)\, ds = \begin{pmatrix} 0 \\ t^2/2 \end{pmatrix}$$

$$D^{-1} \omega_b \phi(b) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \tilde{\phi} = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{D^{-1}b} - \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{D^{-1}\omega_b \phi(b)}\underbrace{\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}}_{\int_a^b \phi^{-1}(s) h(s) ds} +$$

$$+ \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 0 \\ t^2/2 \end{pmatrix}}_{\int_a^t \phi^{-1}(s) h(s) ds}$$

$$\tilde{\phi} = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -3/2 \\ t^2/2 + 1 \end{pmatrix} = \begin{pmatrix} t^2/2 - \frac{3}{2}t + 1 \\ -3/2 \end{pmatrix}$$

which satisfies equation and BC.

Green function

We now deal with

$Lu = f$ $\qquad Lu = u^{(n)} + a_1 u^{(n-1)} \dots + a_n u$

$W_a\, k(u(a)) + W_b\, k(u(b)) = \underline{b}$

wronskian vector $\begin{pmatrix} u \\ u' \\ u^{(n-1)} \end{pmatrix}$

Example $\begin{cases} Lu = u'' \\ u'(0) = 0 \quad u(1) = 1/6 \end{cases}$

$(a = 0 \quad b = 1) \rightarrow W_a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ W_b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

We can convert $Lu = f$ to a first order differential equation system

Call $x_1 = u \quad x_2 = u' \quad x_3 = u'' \quad \dots \quad x_n = u^{(n+1)}$

$$\Rightarrow x_1' = u' = x_2$$
$$x_2' = u'' = x_3$$
$$\vdots$$
$$x_n' = u^{(n)} = -a_1 u^{(n-1)} - a_2 u^{(n-2)} \dots - a_n u + f$$
$$= -a_1 x_n - a_2 x_{n-1} \dots - a_n x_1 + f$$

$$\Rightarrow Lu = f \iff \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ -a_n & -a_{n-1} & \cdots & -a_2 \quad -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ f \end{bmatrix}$$

$$Lu = f \iff x' = Ax + h$$

Thus, let: $\phi = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{pmatrix} = \left(k(u_1) \cdots k(u_n)\right)$

$$D = W_a K_a + W_b K_b \qquad K_a = \left(k(u_1(a)) \cdots k(u_n(a))\right)$$

We are interested in $\tilde{u}$

$\Rightarrow \tilde{u} = e_1^T \tilde{\phi} \Rightarrow$

$$\tilde{u} = e_1^T \int_a^b G(t,s)\, \hat{h}(s)\, ds = \int_a^b g(t,s)\, f(s)\, ds \qquad (\hat{h} = e_n f)$$

But $G(t,s) = \begin{cases} \phi(t) D^{-1} W_a \phi(a) \phi^{-1}(s) & s \le t \\ -\phi(t) D^{-1} W_b \phi(b) \phi^{-1}(s) & s > t \end{cases}$

$$g(t,s) = e_1^T G(t,s)\, e_n$$

$$\Rightarrow \boxed{g(t,s) = \begin{cases} \underline{u}^T D^{-1} W_a K_a K^{-1}(s)\, e_n \\ -\underline{u}^T D^{-1} W_b K_b K^{-1}(s)\, e_n \end{cases}}$$

Properties of Green Matrices / functions

1) They are a formal solution of the homogeneous problem (except at $s=t$)

$$\Rightarrow \left.\begin{array}{l} G' = AG \\ W_a G(a) + W_b G(b) = 0 \end{array}\right\}$$

Indeed

$$G(t,s) = \begin{cases} \phi(t)\, H_+(s) & s \le t \\ \phi(t)\, \underset{\substack{\uparrow \\ \text{these act like constants}}}{H_-(s)} & s > t \end{cases}$$

$$\Rightarrow \quad G' = \begin{cases} \phi'(t)\, H_+(s) = A\,\phi(t)\, H_+(s) \\ \phi'(t)\, H_-(s) = A\,\phi(t)\, H_-(s) \end{cases} = AG$$

$$W_a G(a,s) + W_b G(b,s) = W_a\, \phi(a)\, \underset{\substack{\uparrow \\ \text{recall that } t=a \Rightarrow \underline{\underline{s>t}}}}{H_-(s)} + W_b\, \phi(b)\, H_+(s) =$$

$$= [D - W_b\, \phi(b)]\, H_-(s) + W_b\, \phi(b)\, H_+(s) = D\, H_-(s) + W_b\, \phi(b)\, (H_+(s) - H_-(s))$$

$$= -DD^{-1} W_b\, \phi(b)\, \phi^{-1}(s) + W_b\, \phi(b)\, \phi^{-1}(s) = 0$$

QED

2) $G(t,s)$ is discontinuous at $t=s$

$$\lim_{t\to s^+} G(t,s) = G(s^+,s) = \phi(s^+)\,H_+(s^+)$$

$$\lim_{t\to s^-} G(t,s) = G(s^-,s) = \phi(s^-)\,H_-(s^-)$$

$$\Rightarrow G(s^+,s) - G(s^-,s) = \phi(s)\left[H_+(s^+) - H_-(s^-)\right] =$$

$$= \phi(s)\, D^{-1}\cdot\left[W_a\,\phi(a) + W_b\,\phi(b)\right]\phi^{-1}(s) =$$

$$= I$$

Corolary $K(g(t,s))$ is discontinuous at $t=s$

wronskian vector

$$g(t,s) = \begin{cases} u^T(t)\,h^+(s) \\ u^T(t)\,h_-(s) \end{cases}$$

$$\Rightarrow K(g(t,s)) = \begin{cases} K(u^T)\,h^+(s) \\ K(u^T)\,h_-(s) \end{cases}$$

but $h_+ - h_- = K^{-1}(u(s))\,\underline{e_n}$

$$\Rightarrow k(g(s^+,s)) - k(g(s^-,s)) = K(u(s))\, K^{-1}(u(s))\, \underline{e_n} = \underline{e_n}$$

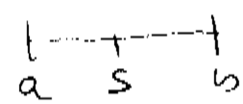
$$\Rightarrow \begin{cases} g(s^+,s) - g(s^-,s) = 0 \\ g'(s^+,s) - g(s^-,s) = 0 \\ \vdots \\ g^{(n-1)}(s^+,s) - g^{(n-1)}(s^-,s) = 1 \end{cases}$$

Determination of green functions knowing their properties

<u>Example</u> $u'' = f$, $u'(0) = 0$ $u(1) = 2$

$$\Rightarrow g(t,s) = A(s) + B(s)t \quad \leftarrow G \text{ is solution}$$

will take $\neq$ values at each side of $s=t$



In $[a,s)$ i.e. $s>t$ $\quad g(t,s) = A_1(s) + B_1(s)t$

In $(s,1] \Rightarrow s<t \quad g(t,s) = A_2(s) + B_2(s)t$

Now g satisfies BC (homogeneous)

$\Rightarrow g'(0,s) = B_1(s) = 0 \qquad u'(0)=0$

$g(1,s) = A_2(s) + B_2(s) = 0 \qquad u(1)=0$

$$\Rightarrow g(t,s) = \begin{cases} A_1(s) & s>t \\ A_2(s)[1-t] & s<t \end{cases}$$

Discontinuity.

$t \to s^+ \Rightarrow t>s$

$g(s^+,s) - g(s^-,s) = 0 \Rightarrow A_1(s) - A_2(s)(1-s) = 0$

$g'(s^+,s) - g'(s^-,s) = 1 \Rightarrow 0 - A_2(s) = 1$

$\Rightarrow A_2(s) = -1$

$A_1(s) = A_2(s)(1-s) = s-1$

$$\Rightarrow g(t,s) = \begin{cases} s-1 & s>t \\ t-1 & s<t \end{cases}$$

$$u = \int_0^1 g(t,s) f(s)\,ds + \underbrace{\varphi^T \phi(t) D^{-1} b}$$