Functions of Complex Variables

Basic Review (left for student, we will just outline it).

Functions:

$$e^z = e^x e^{iy} = e^x\left(1 + (iy) + \frac{1}{2!}(iy)^2 + \dots\right)$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \dots\right) + i$$

$$\left(y - y^3/3! + \dots\right)$$

$$= e^x(\cos y + i \sin y)$$

$$\boxed{e^z = e^x(\cos y + i \sin y)}$$

Euler Formula.

$\cos z$? $\quad e^{-iy} = \cos y - i \sin y$

$\sin z$.

$$\Rightarrow \cos y = \frac{e^{iy} + e^{-iy}}{2}$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2}$$

$$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2}$$

$$\Rightarrow \quad \cos z = \cosh(iz)$$
$$\sin z = \sinh(iz)$$

— o —

<u>Polar forms</u>

$$z = r e^{i\theta}$$

$$z = x + iy$$
$$r = \sqrt{x^2+y^2}$$
$$\theta = \arg(z) = \tan^{-1} y/x$$

Note that $\theta = \arg z = \underline{\mathrm{Arg}}\, z + 2k\pi$

principal argument

$$-\pi \le \mathrm{Arg}\, z \le \pi$$

θ_0

— o —

<u>Moivre's Formula</u>

$$z^n = (r e^{i\theta})^n = r^n e^{in\theta}$$
$$= r^n (\cos n\theta + i \sin n\theta)$$

— o —

Logarithm function:

$$\log z = \log(r e^{i\theta}) = \ln r + \log(e^{i\theta})$$

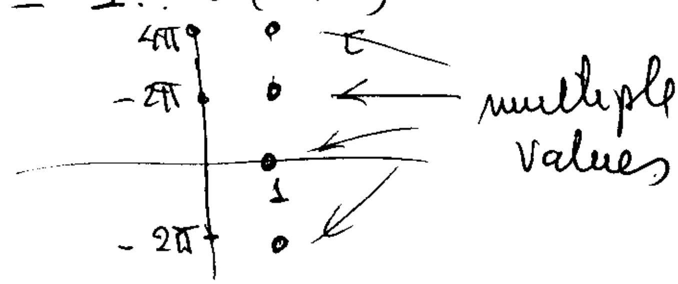
$$= \ln r + i\theta \ln e = \ln r + i\theta$$

$$= \ln r + i(\theta_0 + 2\pi k)$$

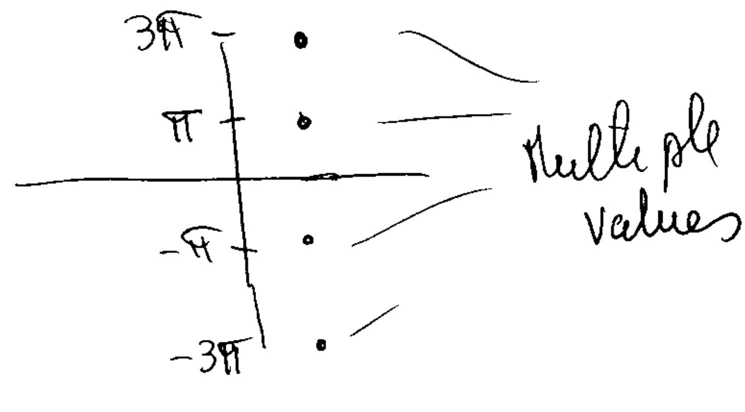
$k = 0, \pm 1, \pm 2 \ldots$

Example.

$$\ln e = 1 + i(2\pi k)$$



$$\ln -e = 1 + i(\pi + 2k\pi)$$



Limits: pretty much the same as in real calculus

$$\lim_{z \to z_0} f(z) = L$$

if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - L| < \varepsilon \quad \forall z \ni |z - z_0| < \delta.$$

If $\left\{ \begin{array}{l} f(z_0) = L \\ \lim_{z \to z_0} f(z) = L \end{array} \right\}$ we say the function is continuous

Differentiability

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

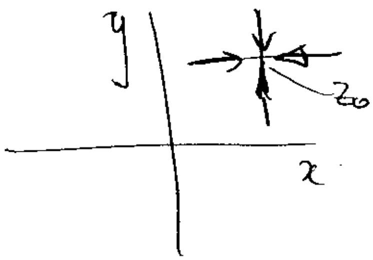
or

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

we need, as usual, the limit to be unique.

let $f(z) = u(x,y) + i\,v(x,y)$

$$\Delta z = \Delta x + i\,\Delta y$$



let $\Delta y = 0$

$\Rightarrow$

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{[u(x_0+\Delta x, y_0) + i\,v(x_0+\Delta x, y_0) - u(x_0,y_0) - i\,v(x_0,y_0)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

let $\Delta x = 0$

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) + i\,v(x_0, y_0+\Delta y) - u_0 - i\,v_0}{i\,\Delta y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

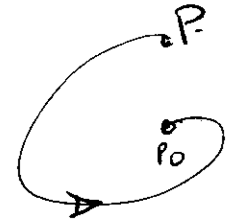
Cauchy-Riemann Equations

Comparing $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Differentiability

- Cauchy Riemann equations are a necessary condition
- Sufficient Conditions: Cauchy Riemann equations + $u, v \in C^1$ in some neighborhood of z_0.

This is proven in Greenbergs by considering an arbitrary approach to P_0



Analytic functions

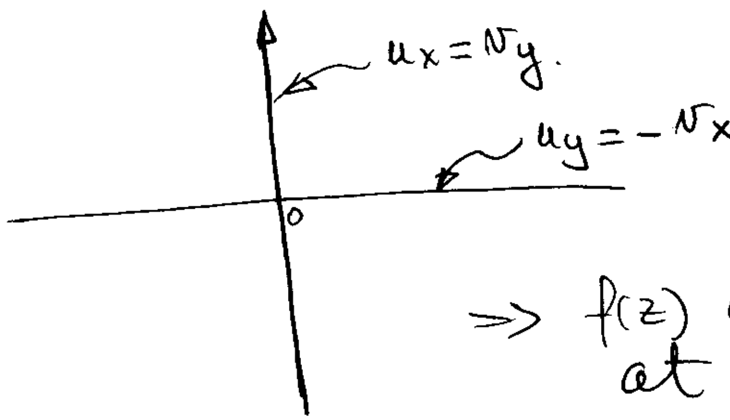
$f(z)$ analytic if it is differentiable at z_0 and ~~at~~ some neighborhood of z_0. Otherwise $f(z)$ is singular at z_0. Analytic functions everywhere are called entire.

Example

$$f(z) = |z|^2 = z\bar{z}$$

$$\Rightarrow f(z) = \underbrace{(x^2+y^2)}_{u} + \underbrace{0}_{v}i.$$

$$\Rightarrow u_x = 2x \qquad u_y = 2y$$
$$v_x = 0 \qquad v_y = 0$$



$\Rightarrow f(z)$ differentiable at $z=0$

Not analytic anywhere

Derivatives: Pretty much the same as in Real Calculus.

$$\frac{d}{dz} e^z = e^z \quad \text{etc.}$$

Harmonic function $f(z)$ is analytic

$\Rightarrow$ u, v are **harmonic** because

$$\nabla^2 u = 0 \quad \nabla^2 v = 0$$