Fourier Series

Taylor Series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Fourier Series $f(x) = \sum_{n=0}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

$f(x)$ is periodic of period $2l$

Let's take a look at the periods

n	functions	Periods
0	1	—
1	$\cos\frac{\pi x}{l}$ $\sin\frac{\pi x}{l}$	$2l, 4l, 6l$
2	$\cos\frac{2\pi x}{l}$ $\sin\frac{2\pi x}{l}$	$l, 2l, 3l$
3	$\cos\frac{3\pi x}{l}$ $\sin\frac{3\pi x}{l}$	$\frac{2l}{3}, \frac{4l}{3}, 2l, \frac{8l}{3}$

$2l$ is the smallest period shared by all terms.

How to obtain coefficients.

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Indeed

$$\int_{-l}^{l} f(x)\,dx = 2l\, a_0 + \sum_{n=0}^{\infty} a_n \overbrace{\int_{-l}^{l} \cos\frac{n\pi}{l}x\,dx}^{=0} + \sum_{n=0}^{\infty} b_n \int_{-l}^{l} \sin\frac{n\pi}{l}x\,dx$$

$$\int_{-l}^{l} f(x)\cos\frac{m\pi}{l}x\,dx = a_0 \overbrace{\int_{-l}^{l} \cos\frac{m\pi}{l}x\,dx}^{=0} + \sum_{n=0}^{\infty} a_n \int_{-l}^{l} \cos^2\frac{m\pi}{l}x\,dx + \sum_{n=0}^{\infty} b_n \int_{-l}^{l} \cos\frac{m\pi}{l}x \sin\frac{n\pi}{l}x\,dx$$

but $\int_{-l}^{l} \cos^2\frac{m\pi}{l}x\,dx = l$

$\int_{-l}^{l} \sin\frac{m\pi}{l}x \cos\frac{n\pi}{l}x\,dx = 0$

Same proof for b_n

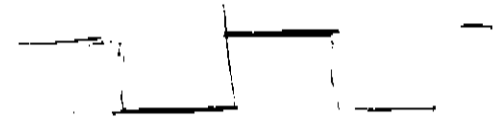
<u>Fourier Convergence Theorem</u>

f periodic with period $2l$,

f, f' piecewise continuous in $[-l, l]$ $\Rightarrow$

$\sum_{n=0}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$ <u>converges</u> to $f(x)$ where $f(x)$ is continuous and to $\frac{f(x^+)+f(x^-)}{2}$ where is discontinuous.

<u>Example</u> $f(x)=\begin{cases}0 & [-l,0)\\ 4 & [0,l]\end{cases}$



$$a_0=\frac{1}{2l}\int_{-l}^{l} f(x)\,dx=\frac{1}{2l}\int_{-l}^{0} 0\,dx+\frac{1}{2l}\int_{0}^{l} 4\,dx=2$$

average of values at $x=0$

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx=$$

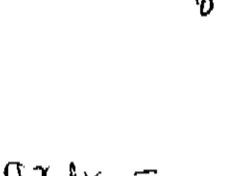
$$=\frac{1}{l}\int_{-l}^{0} 0\,dx+\frac{1}{l}\int_{0}^{l} 4\cos\frac{n\pi x}{l}=\frac{l4}{n\pi}\sin\frac{n\pi x}{l}\Big|_0^l=0$$

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx=\frac{1}{l}\int_{0}^{l} 4\sin\frac{n\pi x}{l}\,dx=$$

$$=-\frac{l4}{n\pi}\left[\cos\frac{n\pi x}{l}\right]_0^l=$$

$$=\frac{4l}{n\pi}\left(1-\cos n\pi\right)$$

$$b_n=\frac{4l}{n\pi}\left[1-(-1)^n\right]=\begin{cases}\frac{8l}{n\pi} & n=1,3\dots\\ 0 & n=2,4\dots\end{cases}$$



first 2 terms

$$\Rightarrow f(x)=2+\frac{8l}{\pi}\sum_{1,3,}\frac{1}{n}\sin\frac{n\pi x}{l}$$

$$=2+\frac{8l}{\pi}\sum_{1}^{\infty}\left(\sin\frac{(2n-1)\pi x}{l}\right)\left(\frac{1}{2n-1}\right)$$

Gibbs phenomenon

What about convergence then? -

answer: the overshoot point moves and convergence is for x fixed.

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Even functions

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \overset{f(x)=f(-x)}{=} \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0$$

Odd functions.

$$a_0 = 0 \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$
$$a_n = 0$$

Sturm Liouville Theory

Sturm Liouville Problem

$$[p(x)y']' + q(x)y + \lambda w(x) y = 0$$

$$\left\{\begin{array}{l} \alpha y(a) + \beta y'(a) = 0 \\ \gamma y(b) + \delta y'(b) = 0 \end{array}\right\} \underline{\underline{w_a k_a + w_b k_b = 0}}$$

$$\left.\begin{array}{ll} p, q, w & \text{continuous} \\ p(x) > 0 & \\ w(x) > 0 & \end{array}\right\} \text{in } [a, b]$$

Any λ that allows a non-trivial solution is called an eigenvalue of the problem

$Ly = \lambda y$ $\qquad$ $Ly = \frac{1}{w(x)}[p(x)y']' + \frac{q(x)}{w(x)} y$

Example: $y'' + \lambda y = 0 \qquad \begin{array}{l} y(0) = 0 \\ y(L) = 0 \end{array}$

$\Rightarrow y(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x.$

$\lambda = 0 \qquad y(x) = A \ ?$ $\quad$ but for $\lambda = 0$ $y'' = 0$ $\Rightarrow \boxed{y = A + Bx}$

$y(0) = 0 \Rightarrow A = 0$

$y(L) = 0 \Rightarrow BL = 0 \Rightarrow B = 0$ | solution is trivial

$\lambda \neq 0 \quad y(0) = A = 0$
$\quad\quad\quad y(L) = 0 = B \sin\sqrt{\lambda} L$

$B \neq 0$ or we have a trivial solution

$\Rightarrow \sin\sqrt{\lambda}L = 0 \quad \Rightarrow \quad \sqrt{\lambda}L = n\pi \quad n = 0, \pm 1, \pm 2.$

Since sin is odd use $n = 0, 1, 2, \ldots$

$$\Rightarrow \quad \lambda_n = \frac{n^2\pi^2}{L^2} \quad \Rightarrow \quad \phi_n = \sin\frac{n\pi x}{L}$$

eigenvalue

Solution (eigenfunction)

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1) Eigenvalues are real
2) " " simple (not repeated rts)
$\Rightarrow$ one eigenfunction per eigenvalue.
3) Eigenfunctions are orthogonal
4) Eigenfunctions can be used as a basis to obtain an expansion series of f (f f' piecewise continuous).

$$f(x) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n(x)$$

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, w(x)\, dx$$

weight function

Periodic BC.

$$\left.\begin{aligned} y(a) &= y(b) \\ y'(a) &= y'(b) \end{aligned}\right\}$$ Does not fall under previous definition.

Singular Cases

$p(x)$ (and/or $w(x)$) vanish at a, or b (end of interval).

$\Rightarrow$ Need to modify BC.

$$\left.\begin{aligned} p(a) &= 0 \\ p(b) &\neq 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &y \text{ bounded at } a \\ &\gamma y(b) + \beta y'(b) = 0 \end{aligned}$$

$$\left.\begin{aligned} p(a) &= 0 \\ p(b) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &y \text{ bounded at } a \\ &y \text{ bounded at } b \end{aligned}$$

Properties.

1) Eigenvalues are real
2) Eigenfunctions are orthogonal.

<u>Example</u> $y'' + \lambda y = 0 \qquad x \in [-L, L]$

$$y(-L) = y(L) \qquad y'(-L) = y'(+L)$$

$$y(x) = \begin{cases} A \cos\sqrt{\lambda}x + B \cos\sqrt{\lambda}x \, . & \lambda \neq 0 \\ C + Dx & \lambda = 0 \end{cases}$$

$\lambda = 0 \qquad \cancel{C} - DL = \cancel{C} + DL$

$\qquad \Rightarrow D = 0 \qquad \Rightarrow$ $\boxed{y(x) = C}$

$\lambda = 0$ is an eigenvalue

$\phi_0 = \underline{1}$ eigenfunction.

$\lambda \neq 0 \qquad \sin\sqrt{\lambda}L \, B = 0$

$\qquad\qquad \sin\sqrt{\lambda}L \, A = 0$

$\Rightarrow \sin\sqrt{\lambda}L = 0 \qquad \lambda = \left(\frac{n\pi}{L}\right)^2$

$$\phi_n = \cos\frac{n\pi x}{L}$$

$$\Rightarrow \quad = \sin\frac{n\pi x}{L}$$

<u>two eigenfunctions</u>

Bessel function example (Singular case)

$$(xy')' + \lambda x y = 0$$

$y(0)$ bounded

$y(L) = 0$

$$y(x) = A\, J_0(\sqrt{\lambda}\, x) + B\, Y_0(\sqrt{\lambda}\, x) \qquad \lambda \neq 0$$

$$y(x) = C + D \ln x \qquad \lambda = 0$$

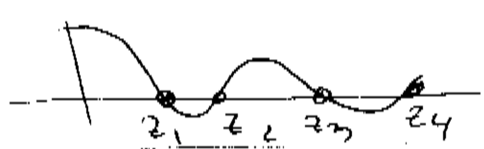
$\lambda = 0$ $\quad y(0)$ bounded $\Rightarrow D = 0$

$y(L) = 0 \Rightarrow C = 0 \quad \Rightarrow \lambda = 0$ not eigenvalue

$\lambda \neq 0 \quad B = 0 \quad Y_0(0)$ is unbounded. $(\to -\infty)$

$$y(L) = A\, J_0(\sqrt{\lambda} L) = 0$$

$\sqrt{\lambda} L$ zero of $J_0(x)$



$$\boxed{\lambda_n = \left(\frac{z_n}{L}\right)^2 \qquad \phi_n = J_0\left(z_n \frac{x}{L}\right)}$$