# Fourier Integral / Transform

Consider $e^{-x^2}$  .

This function is not periodic. $\Rightarrow$ cannot expand in Fourier series!

Now consider $$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$\boxed{\text{* Now let } l \to \infty}$

But we have the $\frac{n\pi}{l}$ term (frequency spectrum)

Observe

| $l$ | $n\pi/l$ |
|---|---|
| $\pi$ | 0, 1, 2, 3, 4 |
| $2\pi$ | 0, 0.5, 1, 1.5, 2 |
| $10\pi$ | 0, 0.1, 0.2, 0.3, 0.4 |

In other words as $l \to \infty$ the spectrum becomes <u>continuous</u>!

Thus

$$f(x) = \overbrace{\int_0^{\infty} \left[ a(\omega) \cos\omega(x) + b(\omega) \sin\omega x \right] d\omega}^{\text{Fourier Integral}}$$

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos\omega x \, dx$$

$$b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin\omega x \, dx$$

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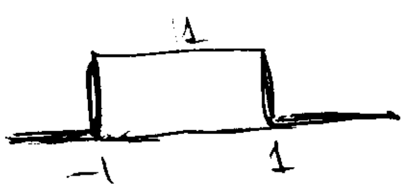
Example: Rectangular pulse



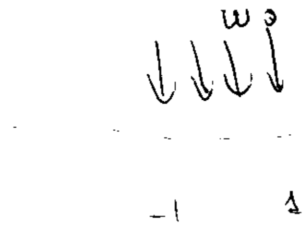

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos\omega x \, dx =$$

$$= \frac{1}{\pi} \int_{-1}^{1} \cos\omega x \, dx = \frac{2}{\pi} \frac{\sin\omega}{\omega}$$

$$b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin\omega x \, dx = 0$$

$$\Rightarrow \boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin\omega}{\omega} \cos\omega x \, d\omega}$$

# Infinite Beam problem




$u$ = deflection $\quad EI\,u'''' + ku = w(x)$

$$w = \begin{cases} w_0 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

$w(x)$ is the rectangular pulse

$$\Rightarrow w(x) = \frac{2w_0}{\pi}\int_0^\infty \frac{\sin w}{w} \cos wx \, dx$$

and $\quad u(x) = \int_0^\infty a(w) \cos wx \, dx$

$b(w) = 0 \leftarrow$ Why?

Now take $\quad EI\,u^{IV} + ku = w(x)$

and replace

$$\frac{du}{dx} = \int_0^\infty a(w)\, w \sin wx \, dx$$

$$\Rightarrow u^{IV} = \int_0^\infty a(w)\, w^4 \cos wx \, dx$$

$$\Rightarrow EI\,u^{IV} + ku = \int_0^\infty (EI\,w^4 + k)\, a(w) \cos wx \, dw$$

$$= w(x) = \frac{2w_0}{\pi}\int_0^\infty \frac{\sin w}{w} \cos wx \, dx$$

$$\Rightarrow (EI\omega^4 + k)a(\omega) = \frac{2w_0}{\pi}\frac{\sin\omega}{\omega}$$

$$\Rightarrow a(\omega) = \frac{2w_0}{\pi}\frac{\sin\omega}{\omega}\frac{1}{(EI\omega^4 + k)}$$

$$\Rightarrow \boxed{u(x) = \frac{2w_0}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\frac{\cos\omega x}{(EI\omega^4 + k)}\,d\omega}$$

what to do with this is another story!

## FOURIER TRANSFORM

$$f(x) = \int_0^\infty [a(\omega)\cos\omega x + b(\omega)\sin\omega x]\,d\omega$$

$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\,d\xi \qquad b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin\omega\xi\,d\xi$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)[\cos\omega\xi\cos\omega x + \sin\omega\xi\sin\omega x]\,d\xi\,d\omega$$

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)\cos\omega(\xi - x)\,d\xi\,d\omega}$$

but $\cos\omega(\xi-x) = \dfrac{e^{i\omega(\xi-x)} + e^{-i\omega(\xi-x)}}{2}$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} \left[e^{i\omega(\xi-x)} + e^{-i\omega(\xi-x)}\right] f(\xi)\,d\xi\,d\omega$$

but $$\int_0^\infty \int_{-\infty}^{\infty} e^{i\omega(\xi-x)} f(\xi)\,d\xi\,d\omega \underset{\theta=-\omega}{=} -\int_0^{-\infty} \int_{-\infty}^{\infty} e^{-i\theta(\xi-x)} f(\xi)\,d\xi\,d\theta$$

$$\underset{\omega'=\theta}{=} \int_{-\infty}^{0} \int_{-\infty}^{\infty} e^{-i\omega'(\xi-x)} f(\xi)\,d\xi\,d\omega'$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega(\xi-x)} f(\xi)\,d\xi\,d\omega$$

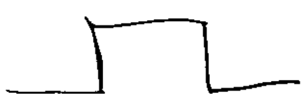
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left\{\int_{-\infty}^{\infty} e^{-i\omega\xi} f(\xi)\,d\xi\right\}}_{f(\omega)\ \rightarrow\ \text{fourier transform.}} d\omega$$

$$F(f) = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\xi} f(\xi)\,d\xi$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\cdot e^{i\omega x}\,d\omega$$

← Inversion formula

Rectangular pulse



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} [H(x+1)-H(x-1)]\,e^{-i\omega x}\,dx$$

$$= \int_{-1}^{1} e^{-i\omega x}\,dx = \left.\frac{e^{-i\omega x}}{-i\omega}\right|_{-1}^{1} = \frac{2\sin\omega}{\omega}$$

$$f(x) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin\omega}{\omega}\, e^{i\omega x}\,d\omega}$$

Need complex analysis to do this!

Properties $F(\alpha f+\beta g) = \alpha F(f) + \beta F(g)$

$$F[f^{(n)}(x)] = (i\omega)^n\cdot \hat{f}(\omega)$$

Convolution

$$F\{f*g\} = \hat{f}(\omega)\,\hat{g}(\omega) \qquad f*g(x) = \int_{-\infty}^{\infty} f(x-\xi)\,g(\xi)\,d\xi$$

Translation

$$F[f(x-a)] = e^{-ia\omega}\,\hat{f}(\omega)$$

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## LAPLACE TRANSFORM

$$\sigma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\sigma(\tau)\,e^{-i\omega\tau}\,d\tau\right]e^{i\omega t}\,d\omega$$

Let
$$\sigma(t) = \begin{cases} e^{-\gamma t} f(t) & t>0 \\ 0 & t<0 \end{cases}$$

Need $\gamma \ni e^{-\gamma t} f(t) \to 0$ as $t \to \infty$

$$\Rightarrow \sigma(t) = H(t)\,e^{-\gamma t} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-\gamma\tau} f(\tau)\,e^{-i\omega\tau}\,d\tau\; e^{i\omega t}\,d\omega$$

Heaviside function

$$\Rightarrow H(t)f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-(\gamma+i\omega)\tau} f(\tau)\,d\tau \cdot e^{(\gamma+i\omega)t}\,d\omega$$

Let $s = \gamma + i\omega$

$$H(t) f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \, e^{st}\, ds$$

integration along a vertical line in the complex plane (we will come to this later)

Thus, the Laplace transform is

$$L\left[f(t)\right] = \bar{f}(s) = \int_0^\infty f(t) e^{-st}\, dt$$

$$L^{-1}\left\{\bar{f}(s)\right\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f(s) e^{st}\, ds = H(t) f(t)$$