Conformal Mapping

Suppose you want to solve

$\nabla^2 \varphi = 0 \Rightarrow \varphi_{xx} + \varphi_{yy} = 0$ in

some domain D.

Let $\begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array}$ be a transformation

such that $D \longrightarrow D'$ where

D' is simpler.

Carry out derivatives $\varphi_x = \varphi_u u_x + \varphi_v v_x$
etc. to obtain.

$$(u_x^2 + u_y^2)\varphi_{uu} + (u_x v_x + u_y v_y)(\varphi_{uv} + \varphi_{vu})$$
$$+ (v_x^2 + v_y^2)\varphi_{vv} + (u_{xx} + u_{yy})\varphi_u +$$
$$+ (v_{xx} + v_{yy})\varphi_v = 0$$

We made the PDE <u>horrible</u>!
but if we require.

1) $u_x = v_y$, $u_y = -v_x$ } Cauchy Riemann relations!

Then the second term of the above equation vanishes.

2) $u, v \in C^2 \Rightarrow$ $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$

$$\Rightarrow u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$$

$\Rightarrow$ We get

$$(u_x^2 + u_y^2)(\varphi_{uu} + \varphi_{vv}) = 0$$

But $u_x^2 + u_y^2 = u_x u_x + u_y u_y =$

$$= \underbrace{u_x v_y - u_y v_x} \neq 0$$

Because $J\left(\frac{u, v}{x, y}\right) \neq 0$

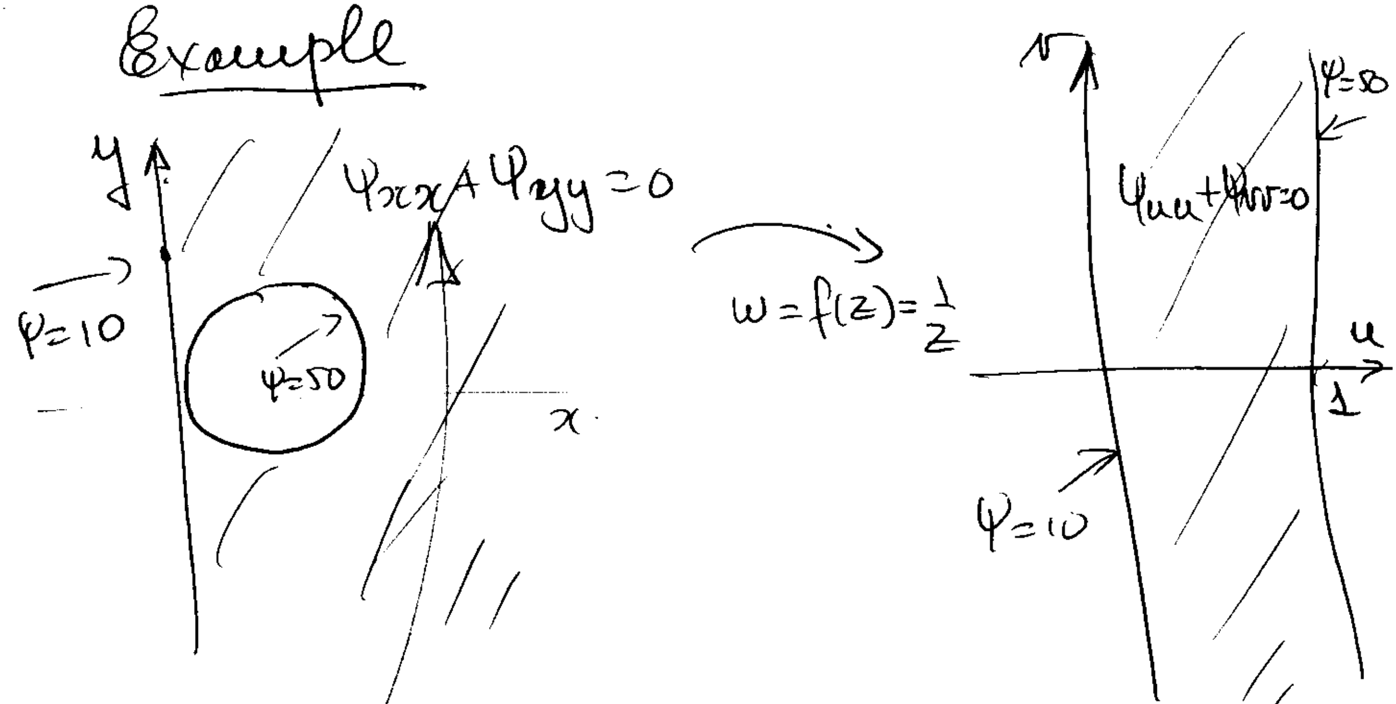
So

3) $J\left(\frac{u, v}{x, y}\right) \neq 0$ (Implicit function theorem $\Rightarrow$ u, v have inverse!!!

$$\Longrightarrow \boxed{\psi_{uu} + \psi_{vv} = 0}$$

in D'

Example



Cannot be solved by separation of variables! Nor by Laplace or Fourier Transform. (Why?)

Fourier transform will work here!

Indeed: $\psi_{uu} + \psi_{vv} = \psi_{uu}$ because there are no BC on v.

$$\Rightarrow \quad \psi = 10 + 40u.$$

$$u + iv = \frac{1}{x+iy} = \frac{x-iy}{x-iy}\,\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow \boxed{u = \frac{x}{x^2+y^2} \qquad v = -\frac{y}{x^2+y^2}}$$

$$\Rightarrow \boxed{\psi = 10 + \frac{40x}{x^2+y^2}}$$

Remarkably simple!!

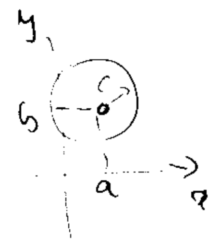
Bilinear Transformation

$$w = \frac{az+b}{cz+d}$$

Analytic everywhere except at $z = -d/c$

$\downarrow$ Image regarding to be at ∞

How do circles map?



let $(x-a)^2+(y-b)^2=c^2$.

let $x=\frac{z+\bar z}{2}$ $\quad y=\frac{z-\bar z}{2i}$

$$\Rightarrow (x-a)^2+(y-b)^2 = x^2+y^2-2ax-2by+a^2+b^2=c^2$$

$$\Rightarrow \quad z\bar z - \underbrace{A}_{=a-ib} z - \bar A \bar z = \underbrace{B}_{c^2-a^2-b^2}$$

let

$z=\frac{1}{w}$.

$$\Rightarrow \quad \frac{1}{w\bar w} - \frac{A}{w} - \frac{\bar A}{\bar w} = B$$

$B\neq 0 \quad \Rightarrow \quad w\bar w + \frac{\bar A}{B}w + \frac{A}{B}\bar w = \frac{1}{B}$

Multiply by $w\bar w$, divide by B

Same form as

$z\bar z - Az - \bar A\bar z = B$!

$\Rightarrow$ CIRCLE in w.

$B = 0$

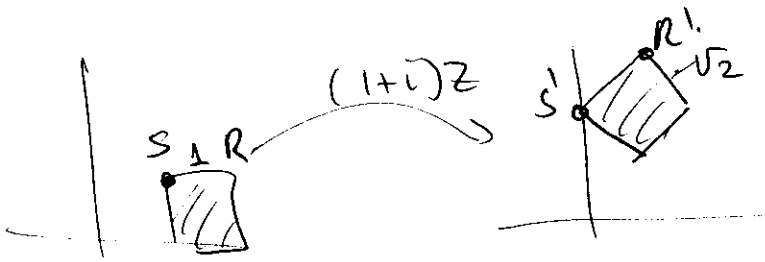
$$1 - A\bar{w} - \bar{A} w = 0$$

$$\left.\begin{array}{l} A = a - ib \\ w = u + iv \end{array}\right\} \Rightarrow \underbrace{1 - 2au + 2bv = 0}_{\text{straight line}}$$

$\Rightarrow \dfrac{1}{z}$ Makes $\left\{ \text{circles} \begin{array}{l} \to \text{circles} \\ \searrow \text{lines} \end{array} \right.$

or circles $\to$ circles
(line considered as a circle with ∞ radius)

<u>Finally</u> $w = \Lambda z$ rotation with scaling



$w = z + B$ translation (self evident)

$w = \dfrac{1}{z}$ inversion