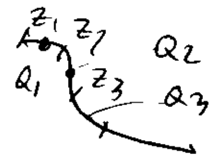
Complex Integral Calculus

Complex Integral

$$I = \int_C f(z)\,dz$$

The integral is the limit of

$$I_n = \sum_{j=1}^{n} f(Q_j)\,\Delta z_j$$



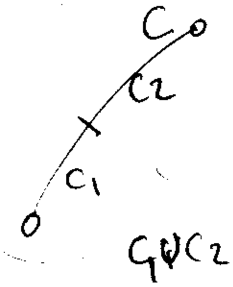
↑ curve in complex plane (piecewise smooth and simple

tangent vector varies continuously)

↑ does not intersect itself

Properties: $\int_C [\alpha f(z) + \beta g(z)]\,dz =$

$$= \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz$$

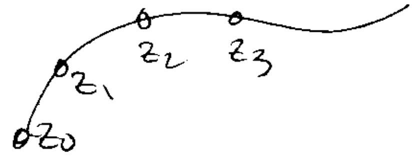


$C_1 \cup C_2 = C$

$$\int_C f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$$

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

$f(z) = z$



$\int_C z\,dz = ?$

$$\Rightarrow I_n^b = z_0(z_1 - z_0) + z_1(z_2 - z_1) + \cdots + z_{n-1}(z_n - z_{n-1})$$

$$I_n^e = z_1(z_1 - z_0) + z_2(z_2 - z_1) + \cdots + z_n(z_n - z_{n-1})$$

$f(Q_i)$ evaluated at beginning

$f(Q_i)$ evaluated at end

$$\Rightarrow I_n^b + I_n^e = z_n^2 - z_0^2$$

$$\Rightarrow I = \int_C z\,dz = \frac{z^2}{2}\Big|_{z_0}^{z_n}$$

coincidence? NOT.

Another approach.

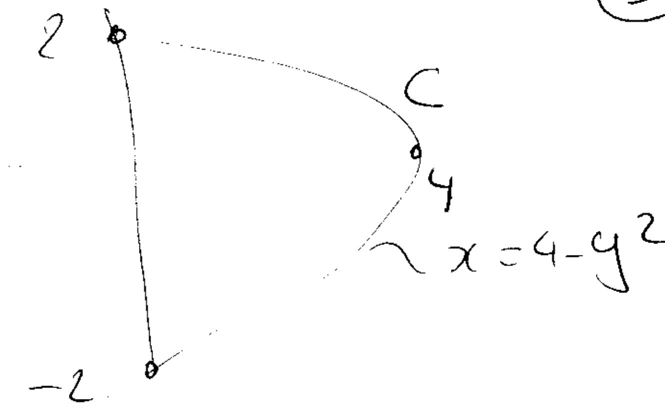
$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy) =$$

$$= \underbrace{\int_C (u\,dx - v\,dy)}_{} + i \underbrace{\int_C (v\,dx + u\,dy)}_{}$$

real line integrals

<u>Example</u>

$$I = \int_C z^2\, dz$$



$$I = \int_C \left[(x^2 - y^2)\,dx - 2xy\,dy \right] + i \int_C (2xy\,dx + (x^2 - y^2)\,dy$$

$$= \int_{-2}^{2} \left[(x^2 - y^2)\frac{dx}{d\tau} - 2xy\frac{dy}{d\tau} \right] d\tau +$$

$$+ i \int_{2}^{-2} \left(2xy\frac{dx}{d\tau} + (x^2 - y^2)\frac{dy}{d\tau} \right) d\tau$$

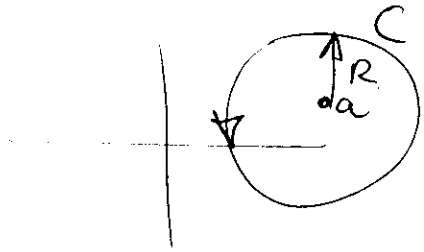
$$= \int_{2}^{-2} \left\{ \left[(4-\tau^2)^2 - \tau^2 \right] \cdot (-2\tau) - 2(4-\tau^2)\tau \right\} d\tau$$

$$+ i \int_{2}^{-2} \left\{ 2(4-\tau^2)\,\tau(-2\tau) + \left[(4-\tau^2)^2 - \tau^2 \right] \right\} d\tau$$

$$= \frac{16}{3} i$$

Use of polar coordinates

$$I = \oint_C (z-a)^n \, dz$$



$$z - a = R e^{i\phi}$$

$$\Rightarrow I = \int_0^{2\pi} (R e^{i\phi})^n \underbrace{\{R i \, e^{i\phi} d\phi\}}_{dz} =$$

$$= i R^{n+1} \int_0^{2\pi} e^{i(n+1)\phi} d\phi = \frac{R^{n+1}}{n+1} e^{i(n+1)\phi} \Big|_0^{2\pi}$$

$$= 0$$

$n \neq -1$

For $n = -1$ $\boxed{I = i R^0 \int_0^{2\pi} e^{i0\phi} d\phi = 2\pi i}$

$-2\pi i$ if curve is traveled clockwise!

Cauchy's Theorem

$$\oint_C f(z)\,dz = 0$$

C: piecewise smoot simple closed curve in D.

f(Z) analytic in D

$$\int_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i \oint_C (u\,dy + v\,dx)$$

$$= \oint \underline{w}\, d\underline{R}$$

$$\nabla \times \underline{w} = \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k} = 0$$

because of Cauchy riemann equations (f is analytic)

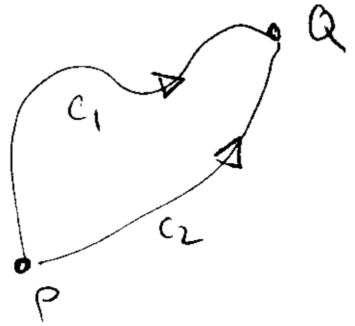
$$d\underline{R} = \underline{i}\,dx + \underline{j}\,dy + \underline{k}\,dz$$

$$\Rightarrow \oint \underline{w}\, d\underline{R} = 0$$

Same thing for $\oint (u\,dy + v\,dx)$.

QED

Path Independence



If $f(z)$ analytic in a simply connected domain D
$\Rightarrow \int_C f(z)\,dz$ is independent of path for any given set of two points.

Proof - $\int_{C_1+(-C_2)} f(z)\,dz = 0$

QED

Fundamental Theorem of Complex Integral Calculus

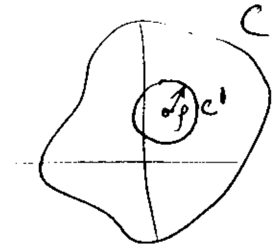
- $G(z) = \int_{z_0}^{z} f(\xi)\,d\xi$ is analytic in D
- $G'(z) = f(z)$
- If $F(z)$ is the primitive of $f(z)$ [$F'(z) = f(z)$

$\Rightarrow \int_{z_0}^{z} f(\xi)\,d\xi = F(z) - F(z_0)$

Thank God!
Nothing new!

Cauchy Integral formula

$$\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a)$$



Proof

$$I = \oint_C \frac{f(z)}{z-a} = \oint_{C'} \frac{f(z)}{z-a}$$

$$= \oint_{C'} \frac{f(a)}{z-a}\,dz + \oint_{C'} \frac{f(z)-f(a)}{z-a}\,dz$$

$$= f(a)\,2\pi i + \underbrace{\oint_{C'} \frac{f(z)-f(a)}{z-a}\,dz}_{J}$$

But $|J| \le \frac{M}{\rho}\, 2\pi\rho = 2\pi M$

Max of $f(z)-f(a)$

This result comes from $\left|\oint f(z)\,dz\right| \le ML$

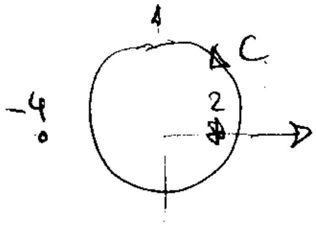
$|J_M| = \left|\sum f(Q)\Delta z\right| \le \sum|\ \ | \le M\sum|\Delta z| \le ML$

But $M \to 0$ as $\rho \to 0$

QED

Examples

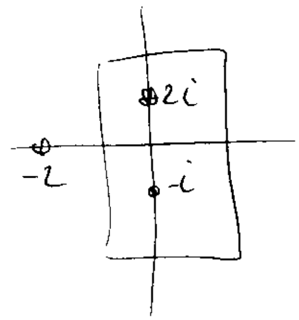
$$I = \oint \frac{e^z}{(z-2)(z+4)}\, dz$$

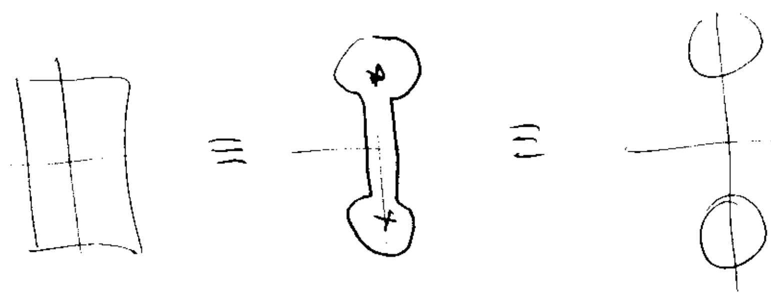


integrand is singular at $\underbrace{z=2}$, $z=-4$ falls outside curve

$$\Rightarrow \quad f(z) = \frac{e^z}{(z+4)}$$

$$I = \oint \frac{\cos z}{(z+2)(z+i)(z-2i)}\, dz$$





Divide and conquer strategy!

Generalized Cauchy Integral

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

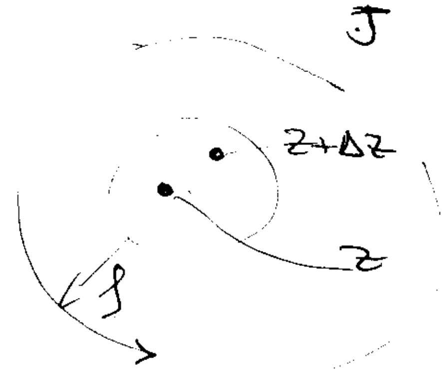
Proof

Let
$$I(z) = \oint_C \frac{f(\xi)}{(\xi - z)}\, dz.$$

$$I'(z) = \lim_{\Delta z \to 0} \oint_C \frac{f(\xi)}{\Delta z}\left(\frac{1}{\xi - z - \Delta z} - \frac{1}{\xi - z}\right) d$$

$$= \lim_{\Delta z \to 0} \oint \frac{f(\xi)\, d\xi}{(\xi - z)(\xi - z - \Delta z)}$$

$$= \lim_{\Delta z \to 0} \left\{ \oint_C \frac{f(\xi)\, d\xi}{(\xi - z)^2} + \Delta z \underbrace{\oint \frac{f(\xi)\, d\xi}{(\xi - z)^2 (\xi - z - \Delta z)}}_{J} \right\}$$



$$|J| \le |\Delta z| \cdot M \; 2\pi \rho$$

Max of $\dfrac{f(\xi)}{(\xi - z)^2 (\xi - z - \Delta z)}$

$$I'(z) = \oint \frac{f(\xi)\,d\xi}{(\xi - z)^2}$$

But $I(z) = 2\pi i\, f(z)$

$$\Rightarrow I'(z) = 2\pi i\, f'(z)$$

$$\Rightarrow \oint \frac{f(\xi)\,d\xi}{(\xi - z)^2} = 2\pi i\, f'(z)$$

We proved it for $n=1$.

Can use induction now.

Assume $\oint \frac{f(\xi)\,d\xi}{(\xi - z)^n} = \frac{2\pi i}{n!} f^{(n)}(z)$ is true $\Rightarrow$ Prove that it is true for $(n+1)$